Thèse de Doctorat de l'Université Paris VI – Pierre et Marie Curie

Spécialité :

Mathématiques

présentée par

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pour obtenir le grade de Docteur de l'Université Paris VI – Pierre et Marie Curie

Quelques problèmes de géométrie complexe et presque complexe

Soutenue le 19 octobre 2009 devant le jury composé de :

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Remerciements

Je remercie Claire Voisin d'avoir accepté de diriger ma thèse. Pendant ces trois années, j'ai pu apprécier sa très grande rigueur intellectuelle, et aussi la variété de ses points de vue, conciliant une connaissance parfaite de la géométrie algébrique complexe et des approches très enrichissantes issues entre autres de la topologie algébrique et de la géométrie différentielle. En allant directement au cœur des problèmes, en distinguant avec une grande acuité ce qui reste à inventer de ce qui peut simplement être perfectionné, elle m'a montré ce que doit être la recherche.

Je n'aurais pas pu mener à bien ce travail sans son optimisme sans faille, sa hauteur de vue, sa gentillesse et sa disponibilité. Je lui en suis très profondément reconnaissant.

Je remercie très sincèrement Philippe Eyssidieux et Daniel Huybrechts qui ont accepté d'être les rapporteurs de ma thèse. Je remercie Christian Peskine, Pierre Schapira et Christophe Sorger, qui me font l'honneur de participer au jury.

J'ai effectué ma thèse à l'Institut de Mathématiques de Jussieu, et je remercie Gilles Godefroy pour son soutien, ses conseils, et l'aide efficace qu'il m'a apportée dans mes démarches administratives. Je remercie aussi tous les mathématiciens avec qui j'ai pu avoir des discussions fructueuses, en particulier Pierre Schapira et Thierry de Pauw.

J'ai fait partie pendant trois ans de l'équipe de Topologie et Géométrie Algébriques. Je remercie les membres de cette équipe et son directeur, Vincent Maillot, qui a toujours eu à cœur de résoudre mes problèmes matériels.

Je remercie tous les professeurs qui m'ont appris des mathématiques tout au long de mes études, et particulièrement le premier d'entre eux, Joseph Oesterlé, à qui je dois beaucoup.

Je remercie enfin ma famille pour sa compréhension et son soutien constant.

Résumé

Le travail effectué dans cette thèse consiste à construire et adapter dans d'autres cadres des objets issus de la géométrie algébrique.

Nous nous intéressons d'abord à la théorie des classes de Chern pour les faisceaux cohérents. Sur les variétés projectives, elle est complètement achevée dans les anneaux de Chow grâce à l'existence de résolutions globales localement libres et se ramène formellement à la théorie pour les fibrés. Un résultat de Voisin montre que ces résolutions n'existent pas toujours sur des variétés complexes compactes générales. Nous construisons ici par récurrence sur la dimension de la variété de base des classes de Chern en cohomologie de Deligne rationnelle pour les faisceaux analytiques cohérents en imposant la formule de Grothendieck-Riemann-Roch pour les immersions et en utilisant des méthodes de dévissage. Ces classes sont les seules à vérifier la formule de fonctorialité par pull-back, la formule de Whitney et la formule de Grothendieck-Riemann-Roch pour les immersions ; elles coïncident donc avec les classes topologiques et les classes d'Atiyah. Elles vérifient aussi le théorème de Grothendieck-Riemann-Roch pour les morphismes projectifs.

Notre second travail est l'étude des schémas de Hilbert ponctuels d'une variété symplectique ou presque complexe de dimension 4. Ils ont été construits par Voisin et généralisent les schémas de Hilbert connus pour les surfaces projectives. En utilisant les structures complexes relatives intégrables introduites dans la construction de Voisin, nous pouvons étendre au cas presque complexe ou symplectique la théorie classique. Nous calculons les nombres de Betti, nous construisons les opérateurs de Nakajima, nous étudions l'anneau de cohomologie et la classe de cobordisme de ces schémas de Hilbert, et nous prouvons dans ce contexte un cas particulier de la conjecture de la résolution crêpante de Ruan.

Mots-clefs

Variétés complexes, Faisceaux cohérents, Classes de Chern, K-théorie analytique, Théorème de Grothendieck-Riemann-Roch, Cohomologie de Deligne.

Schémas de Hilbert ponctuels, Opérateurs de Nakajima, Géométrie presque complexe, Variétés de dimension 4, Orbifolds, Conjecture de la résolution crêpante de Ruan.

Some problems in complex and almost-complex geometry

Abstract

In our thesis, we construct or adapt in other settings notions coming from algebraic geometry.

We first concern ourselves with the theory of Chern classes for coherent sheaves. For projective manifolds, it is complete in the Chow rings via the existence of global locally free resolutions and it is a formal consequence of the theory for algebraic bundles. A result of Voisin shows that these resolutions do not always exist on general complex compact manifolds. We construct here Chern classes in rational Deligne cohomology for coherent analytic sheaves by induction on the dimension of the base manifold. To do so, we prescribe the Grothendieck-Riemann-Roch formula for immersions and we use dévissage methods. The classes we obtain are the only ones which verify the functoriality formula under pullback, the Whitney formula and the Grothendieck-Riemann-Roch formula for immersions; they then coincide with the topological classes and the Atiyah classes. Moreover, they satisfy the Grothendieck-Riemann-Roch theorem for projective morphisms.

The second part of our work consists in studying the punctual Hilbert schemes of a symplectic or almost-complex fourfold. These manifolds have been built by Voisin and generalize the already known Hilbert schemes on projective surfaces. Using the relative integrable structures introduced in Voisin's construction, we can extend the classical theory to the symplectic or almost-complex setting. We compute the Betti numbers, define Nakajima operators, study the cohomology ring and the cobordism class of these Hilbert schemes, and we prove in this context a particular case of Ruan's crepant resolution conjecture.

Keywords

Complex manifolds, Coherent analytic sheaves, Chern classes, Analytic K-theory, Grothendieck-Riemann-Roch theorem, Deligne cohomology.

Puncutal Hilbert schemes, Nakajima operators, Almost-complex geometry, 4-manifolds, Orbifolds, Ruan's crepant resolution conjecture.

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Table des matières

CHAPITRE 1

Introduction

1. Version française

La thèse qui est présentée ici se compose de deux articles, intitulés « Classes de Chern en cohomologie de Deligne pour les faisceaux analytiques cohérents » et « Propriétés topologiques des schémas de Hilbert ponctuels des variétés symplectiques de dimension 4 ». Ces deux sujets relèvent en fait d'une même problématique. Il s'agit de construire et d'adapter des objets issus de la géométrie algébrique dans d'autres cadres : les variétés complexes non projectives dans le premier cas et la géométrie presque complexe et symplectique dans le second.

1.1. Classes de Chern en cohomologie de Deligne pour les faisceaux cohérents.

Dans le premier article nous nous intéressons aux faisceaux analytiques cohérents et aux classes de cohomologie qu'on peut leur associer. La théorie des faisceaux cohérents s'est développée simultanément dans les domaines algébrique et analytique [**Se 1**], [**Ca**], [**Se 2**], [**Ca-Se**], [**Gr-Re**], [**Gr-Ri**]. Sur une variété projective complexe, selon le principe GAGA de Serre [**Se 2**], la catégorie des faisceaux algébriques cohérents est équivalente à la catégorie des faisceaux analytiques cohérents, il n'y a donc pas lieu de distinguer les deux théories. En général, il se peut que des résultats importants se formulent de la même façon pour des faisceaux cohérents algébriques ou analytiques. Cependant les démonstrations sont souvent très différentes. On peut citer trois exemples de ce phénomène.

Le premier est le théorème de finitude qui s'énonce de la manière suivante :

Théorème 1.1. *Si X est une variété projective complexe (resp. complexe compacte) et si $\mathcal{F}$ est un faisceau algébrique (resp. analytique) cohérent sur X, les $H^i(X, \mathcal{F})$ sont des espaces vectoriels de dimension finie.*

La preuve pour les faisceaux algébriques cohérents se réduit par dévissage à l'étude du cas particulier $X = \mathbb{P}^n$ et $\mathcal{F} = \mathcal{O}(k)$. Pour des faisceaux analytiques cohérents sur une variété complexe, la démonstration, due à Cartan et Serre [**Ca-Se**], [**Gr-Re**], utilise des méthodes d'analyse.

Ce théorème admet une version relative :

Théorème 1.2. *Soient X et Y des variétés algébriques (resp. complexes), $f : X \longrightarrow Y$ un morphisme algébrique (resp. holomorphe) et $\mathcal{F}$ un faisceau algébrique (resp. analytique) cohérent sur X. Si f est propre sur le support de $\mathcal{F}$, $f_*\mathcal{F}$ ainsi que les images directes supérieures $R^i f_*\mathcal{F}$ sont cohérents sur Y.*

Dans le cas algébrique, on procède à nouveau par dévissage [**Bo-Se**]. La preuve de Grauert et Riemenschneider [**Gr-Ri**] pour les variétés complexes, beaucoup plus délicate, utilise des théorèmes de finitude pour les espaces de Fréchet nucléaires.

On peut prendre pour troisième exemple l'un des résultats les plus importants de la théorie des faisceaux algébriques cohérents : le théorème de Grothendieck-Riemann-Roch.

THÉORÈME 1.3. *Soient X et Y des variétés quasi-projectives lisses, $f : X \longrightarrow Y$ un morphisme propre et $\mathcal{F}$ un faisceau algébrique cohérent sur X. Alors l'identité suivante est vérifiée dans l'anneau de Chow de Y :*

$$f_*\big(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(X)\big) = \sum_{i\geq 0}(-1)^i\,\mathrm{ch}\big(R^i f_*(\mathcal{F})\big)\,\mathrm{td}(Y).$$

Lorsqu'on prend pour X une variété projective et Y la variété réduite à un point, on obtient la formule de Riemann-Roch-Hirzebruch, initialement établie par des méthodes de cobordisme [**Hirz**] :

$$\chi(X, \mathcal{F}) = \int_X \mathrm{ch}(\mathcal{F})\,\mathrm{td}(X).$$

Si $\mathcal{E}$ est un faisceau localement libre sur une variété complexe compacte X, la formule de Riemann-Roch-Hirzebruch est encore valable pour $\mathcal{E}$ et résulte de la formule d'Atiyah-Singer qui calcule l'indice d'un opérateur elliptique, dans ce cas l'opérateur $\overline{\partial}$. Plus généralement, Toledo et Tong ont montré cette formule pour un faisceau analytique cohérent arbitraire en construisant un parametrix explicite pour l'opérateur $\overline{\partial}$ [**To-To 1**].

Lorsque X et Y sont des variétés complexes compactes et que les classes de Chern sont choisies dans les anneaux de Hodge $\bigoplus_i H^{\,i}\big(X, \Omega_X^i\big)$ et $\bigoplus_i H^{\,i}\big(Y, \Omega_Y^i\big)$ de X et Y, l'égalité de Grothendieck-Riemann-Roch reste toujours vraie : la démonstration en a été faite par O'Brian-Toledo-Tong [**OB-To-To**]. Il reste cependant des situations où le théorème de Grothendieck-Riemann-Roch n'est toujours pas établi, par exemple lorsque les variétés X et Y ne sont pas Kähleriennes et que les classes de Chern sont choisies dans $H^*(X, \mathbb{Q})$ et $H^*(Y, \mathbb{Q})$.

La théorie des classes de Chern est intéressante à comparer dans les cadres algébrique et analytique. Pour les faisceaux localement libres, ces classes peuvent toujours être construites dans les deux cadres par le principe de scindage de Grothendieck [**Grot**], [**Vo 3**]. Les anneaux de cohomologie choisis sur les variétés algébriques sont les anneaux de Chow et ce sont les anneaux les plus généraux possibles, alors que sur des variétés complexes non projectives la meilleure théorie cohomologique connue est celle de Deligne-Beilinson [**Es-Vi**], [**Vo 3**].

La théorie devient très différente pour les faisceaux cohérents. Sur les variétés algébriques, la construction des classes de Chern est complètement achevée : elle est établie dans les anneaux de Chow et se ramène, via l'existence de résolutions globales localement libres [**Bo-Se**], à la théorie pour les faisceaux localement libres. Le contre-exemple suivant de Voisin interdit de transposer cette méthode aux variétés complexes.

THÉORÈME 1.4. [**Vo 4**] *Sur un tore générique complexe de dimension au moins* 3, *le faisceau d'idéaux d'un point n'admet pas de résolution localement libre globale.*

Avant notre travail, plusieurs théories de classes de Chern existaient déjà dans différents anneaux de cohomologie spécifiques : les classes d'Atiyah-Hirzebruch dans $H^*(X, \mathbb{Z})$ [**At-Hi**], celles d'Atiyah dans $\bigoplus_i H^i\big(X, \Omega_X^i\big)$ [**At**] et celles de Green dans $\bigoplus_i \mathbb{H}^{2i}\big(X, \Omega_X^{\bullet \geqslant i}\big)$ [**Gre**], [**To-To 2**].

Notre but dans « Classes de Chern en cohomologie de Deligne pour les faisceaux analytiques cohérents » est de construire des classes de Chern unifiées pour des faisceaux analytiques cohérents sur une variété complexe compacte, à valeurs dans la cohomologie de Deligne rationnelle de la variété de base. Nous développons pour cela une nouvelle approche.

Le contre exemple de Voisin [**Vo 4**] montre que les variétés complexes possèdent en général beaucoup trop peu de fibrés holomorphes pour qu'on puisse espérer contrôler un faisceau cohérent arbitraire à l'aide de tels fibrés.

Parmi les faisceaux cohérents figurent les faisceaux localement libres et, à l'opposé, les faisceaux de torsion : ce sont les faisceaux cohérents supportés par des sous-ensembles analytiques propres de la variété ambiante X. Si Z est une sous-variété lisse de X et $\mathcal{F}$ un faisceau de torsion supporté sur Z de la forme $i_{Z*}\mathcal{G}$ où $\mathcal{G}$ est cohérent sur Z, la formule de Grothendieck-Riemann-Roch pour l'injection $i_Z\colon Z \longrightarrow X$ prédit l'égalité

$$\mathrm{ch}(\mathcal{F}) = i_{Z*}\big(\mathrm{ch}(\mathcal{G})\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\big)$$

et permet donc *a priori* de définir $\mathrm{ch}(\mathcal{F})$ par cette formule si $\mathrm{ch}(\mathcal{G})$ est connu. Cette observation nous conduit à effectuer la construction des classes de Chern par récurrence sur la dimension de X.

La formule de Grothendieck-Riemann-Roch permet ainsi de définir les classes de Chern pour une classe spécifique de faisceaux de torsion. Un premier problème se pose, car il n'est pas clair que la définition de $\mathrm{ch}(\mathcal{F})$ est indépendante de Z. D'autre part, le support d'un faisceau de torsion n'est pas a priori contenu dans une sous-variété lisse de X. La construction des classes de Chern d'un faisceau de torsion général ne peut donc pas s'effectuer directement sur X. Il apparaît nécessaire de changer de base et de raisonner sur des modèles biméromorphes de X. Le théorème de désingularisation d'Hironaka permet de résoudre ces problèmes et de bâtir une théorie cohérente des classes de Chern pour les faisceaux de torsion.

Maintenant, si $\mathcal{F}$ est un faisceau cohérent quelconque et $\mathcal{F}_{\mathrm{tor}} \subseteq \mathcal{F}$ est le sous-faisceau des éléments de torsion, on dira que $\mathcal{F}$ est *localement libre modulo torsion* si $\mathcal{F}/_{\mathcal{F}_{\mathrm{tor}}}$ est localement libre. Si $\mathcal{F}$ est localement libre modulo torsion, la suite exacte

$$0 \longrightarrow \mathcal{F}_{\mathrm{tor}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/_{\mathcal{F}_{\mathrm{tor}}} \longrightarrow 0$$

conduit à définir $\mathrm{ch}(\mathcal{F})$ par la relation

$$\mathrm{ch}(\mathcal{F}) = \mathrm{ch}\big(\mathcal{F}_{\mathrm{tor}}\big) + \mathrm{ch}\Big[\mathcal{F}/_{\mathcal{F}_{\mathrm{tor}}}\Big].$$

Cependant, un faisceau cohérent n'est pas toujours localement libre modulo torsion. Par exemple, si Z est un sous-ensemble analytique de X de codimension au moins 2, le faisceau d'idéaux $\mathcal{J}_Z$ est sans torsion, sans être localement libre. Pour définir les classes de Chern d'un faisceau cohérent général de rang générique strictement positif, on utilise, et c'est un point clé de la construction, le théorème de désingularisation suivant :

THÉORÈME 1.5. [**Ro**] *Soit $\mathcal{F}$ un faisceau cohérent sur X de rang générique r. Il existe alors un morphisme biméromorphe $\pi : \widetilde{X} \longrightarrow X$ obtenu par une suite finie d'éclatements de centres lisses, et un faisceau localement libre $\mathcal{Q}$ sur $\widetilde{X}$ de rang r, tels que l'on ait une surjection $\pi^*\mathcal{F} \longrightarrow \mathcal{Q}$.*

Ce théorème peut s'énoncer de manière intuitive sous la forme :

Si $\mathcal{F}$ est un faisceau cohérent, on peut par une succession d'éclatements convertir toute la singularité de $\mathcal{F}$ en torsion.

Ce résultat est une conséquence immédiate du théorème de platification d'Hironaka [**Hiro**]. La preuve antérieure de [**Ro**] repose uniquement sur le théorème de désingularisation d'Hironaka.

À l'aide des arguments exposés ci-dessus, nous pouvons définir $\mathrm{ch}(\mathcal{F})$ pour tout faisceau cohérent $\mathcal{F}$ sur une variété complexe de dimension n lorsque la théorie est connue pour les variétés de dimension au plus $n-1$. Les classes ainsi construites vérifient par définition la formule de Grothendieck-Riemann-Roch pour l'immersion d'un diviseur lisse.

La récurrence impose de vérifier pour les classes construites un certain nombre de propriétés telles que la fonctorialité par pull-back, la formule du produit et la formule d'additivité de Whitney. Cette dernière formule signifie que pour toute suite exacte

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

de faisceaux cohérents, $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$; elle n'est pas une conséquence formelle de la construction. En effet, si X est projective et si $\mathcal{F}$ est un faisceau cohérent sur X de la forme $i_{Z*}\mathcal{G}$ où Z est une hypersurface lisse de X, notre construction définit $\mathrm{ch}\big(i_{Z*}\mathcal{G}\big)$ en imposant la formule de Grothendieck-Riemann-Roch. D'autre part $\mathcal{F}$ admet une résolution globale localement libre. Si la formule de Whitney est vérifiée, cela signifie qu'on a établi la formule de Grothendieck-Riemann-Roch pour $i_{Z*}\mathcal{G}$, ce qui ne saurait être évident.

Nous obtenons finalement le résultat suivant :

THÉORÈME 1.6. *Soit X une variété complexe compacte. Pour tout faisceau cohérent $\mathcal{F}$ sur X, on peut définir un caractère de Chern $\mathrm{ch}(\mathcal{F})$ dans $\oplus_p H^{2p}_{\mathrm{Del}}(X, \mathbb{Q}(p))$ tel que*

(i) *Pour toute suite exacte $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ de faisceaux cohérents sur X, on a $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$. Le caractère de Chern $\mathrm{ch}\colon K(X) \longrightarrow H^{*}_{\mathrm{Del}}(X, \mathbb{Q})$ est un morphisme d'anneaux.*

(ii) *Si $f\colon X \longrightarrow Y$ est holomorphe et si y appartient à $K(Y)$, alors $\mathrm{ch}(f^{\dagger}y) = f^{*}\,\mathrm{ch}(y)$.* [1]

(iii) *Si $\mathcal{E}$ est un faisceau localement libre, $\mathrm{ch}(\mathcal{E})$ est le caractère de Chern usuel en cohomologie de Deligne rationnelle obtenu par le principe de scindage de Grothendieck.*

(iv) *Si Z est une sous-variété de X lisse et fermée et si $\mathcal{F}$ est un faisceau cohérent sur Z, alors $\big(i_Z, \mathcal{F}\big)$ vérifie le théorème de Grothendieck-Riemann-Roch, c'est-à-dire*

$$\mathrm{ch}\big(i_{Z*}\mathcal{F}\big) = i_{Z*}\Big(\mathrm{ch}(\mathcal{F})\ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big).$$

De par leur construction même, les classes obtenues vérifient le théorème de Grothendieck-Riemann-Roch pour l'immersion d'une hypersurface lisse. Par éclatement, on en déduit le théorème de Grothendieck-Riemann-Roch pour l'immersion d'une sous-variété lisse de codimension quelconque.

La forme la plus générale du théorème de Grothendieck-Riemann-Roch que nous obtenons est la suivante :

THÉORÈME 1.7. *Le théorème de Grothendieck-Riemann-Roch est valable en cohomologie de Deligne rationnelle pour des morphismes projectifs entre variétés complexes compactes.*

[1] La notation $f^{\dagger}$ est définie page 46.

Le résultat s'obtient en décomposant le morphisme f en une immersion $i\colon X \longrightarrow Y \times \mathbb{P}^N$ et la projection $pr_1\colon Y \times \mathbb{P}^N \longrightarrow Y$. Le théorème de Grothendieck-Riemann-Roch pour i est déjà établi. Pour la projection pr_1, l'article [**Bei**] permet de se ramener au cas où Y est réduit à un point.

Remarquons que si les variétés sont projectives, le Théorème 1.7 est la formule de Grothendieck-Riemann-Roch classique de [**Bo-Se**]. Les arguments que nous avons développés sont valables dans les anneaux de Chow pour les variétés projectives et fournissent donc une nouvelle preuve du théorème de Grothendieck-Riemann-Roch dans le cas algébrique.

Il est maintenant naturel d'examiner l'unicité des classes de Chern ainsi construites. Nous montrons un résultat d'unicité général.

Considérons une théorie cohomologique A, c'est-à-dire un foncteur contravariant $X \longrightarrow A(X)$ défini sur les variétés complexes compactes à valeurs dans la catégorie des $\mathbb{Q}$-algèbres graduées et satisfaisant les propriétés suivantes :

(α) Si σ est l'éclatement d'une variété complexe compacte lisse le long d'une sous variété lisse, σ^* est injectif.

(β) Si E est un fibré vectoriel holomorphe sur X et $\pi\colon \mathbb{P}(E) \longrightarrow X$ est le fibré projectif associé, π^* est injectif.

(γ) Si X est une variété complexe compacte lisse et si Y est une sous-variété lisse de codimension d, on dispose d'un morphisme de Gysin $i_*\colon A^*(Y) \longrightarrow A^{*+d}(X)$.

Le théorème d'unicité s'énonce alors :

Théorème 1.8. *Supposons que l'on ait deux caractères de Chern* ch *et* ch$'$ *pour les faisceaux cohérents sur les variétés complexes compactes lisses, à valeurs dans une théorie cohomologique* A *vérifiant* (α), (β) *et* (γ), *tels que :*

(i) ch *et* ch$'$ *vérifient la formule de Whitney,*

(ii) ch *et* ch$'$ *vérifient la formule de fonctorialité,*

(iii) *si* L *est un fibré en droites holomorphe, alors* $\mathrm{ch}(L) = \mathrm{ch}'(L)$,

(iv) *dans les deux théories, la formule de Grothendieck-Riemann-Roch est valable pour les immersions.*

Alors pour tout faisceau cohérent $\mathcal{F}$, $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}'(\mathcal{F})$.

Les classes de Chern déjà construites dans des cohomologies plus faibles que la cohomologie de Deligne vérifient la formule de Whitney et la formule de fonctorialité. Si ces classes vérifient la formule de Grothendieck-Riemann-Roch pour les immersions, elles coïncident donc avec les classes que nous avons construites, et réciproquement.

On obtient le théorème de comparaison suivant :

Théorème 1.9. *Soit* $\mathcal{F}$ *un faisceau analytique cohérent sur* X. *Alors :*

(i) *Les classes* $c_i(\mathcal{F})$ *dans* $H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i))$ *et* $c_i(\mathcal{F})^{\mathrm{top}}$ *dans* $H^{2i}(X, \mathbb{Z})$ *ont la même image dans* $H^{2i}(X, \mathbb{Q})$.

(ii) *L'image de* $c_i(\mathcal{F})$ *par le morphisme naturel de* $H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i))$ *dans* $H^i(X, \Omega^i_X)$ *est la* i-ème *classe d'Atiyah-Chern de* $\mathcal{F}$.

La formule de Grothendieck-Riemann-Roch pour les immersions a été prouvée par Atiyah et Hirzebruch pour les classes topologiques dans [**At-Hi**] et par Toledo et Tong pour les classes d'Atiyah dans [**OB-To-To**].

Même si des raisons de complexité calculatoire ont conduit à privilégier les classes de Chern exponentielles, les résultats semblent pouvoir s'étendre à la cohomologie de Deligne entière, et donc prendre en compte les phénomèmes de torsion.

Le résultat qui nous manque encore est la formule de Grothendieck-Riemann-Roch pour les morphismes arbitraires entre variétés complexes compactes. Le problème est certainement difficile, puisqu'il est toujours ouvert dans le cas particulier de la cohomologie de Betti rationnelle sur les variétés non Kähleriennes (dans le cas des variétés Kähleriennes, comme les classes de Chern en cohomologie de Dolbeault et de Betti rationnelle sont compatibles via la décomposition de Hodge, le résultat est une conséquence de [**OB-To-To**]).

1.2. Topologie des schémas de Hilbert presque complexes et symplectiques.

Notre deuxième travail se place dans le cadre de la géométrie presque complexe. Cette géométrie est d'abord apparue comme une extension de la géométrie complexe. Une structure presque complexe sur une variété différentiable M est la donnée d'un endomorphisme du fibré tangent TM de carré $-\,\mathrm{id}$; elle est donc plus facile à manipuler qu'une structure intégrable, qui se définit par cartes locales. D'après le théorème fondamental suivant

THÉORÈME 1.10. *Sur une variété de dimension deux, une structure presque complexe est toujours intégrable*

on peut en dimension 2 paramétrer les structures complexes de manière remarquablement simple. C'est ce qui constitue l'un des points de départ de la théorie de Teichmüller. Sur des variétés de dimension paire supérieure à 2, les structures presque complexes non intégrables apparaissent effectivement. On peut alors donner une condition nécessaire et suffisante pour qu'une telle structure soit intégrable : c'est le théorème de Newlander-Nirenberg.

THÉORÈME 1.11. *Soit M une variété $\mathcal{C}^{\infty}$ munie d'une structure presque complexe J. Alors J est intégrable si et seulement si $\big[T^{1,0}M, T^{1,0}M\big] \subset T^{1,0}M$.*

Les structures presque complexes constituent un bon espace de paramètres dans lequel on peut étudier, à l'aide de ce théorème, les structures qui sont intégrables. Cette approche est au cœur de la théorie des déformations de Kodaira [**Kod**]. Dans cette théorie, le théorème de Kuranishi paramétrise par un sous-ensemble analytique de l'espace des structures presque complexes les structures intégrables proches d'une structure intégrable initiale fixée.

Les structures presque complexes interviennent là comme espace de paramètres, mais leurs propriétés géométriques propres ne sont pas exploitées. L'approche change avec Gromov, qui n'utilise pas ces structures sur des variétés complexes, mais sur des variétés symplectiques [**Grom**].

La notion utile pour cette étude est celle de structure presque complexe adaptée à une forme symplectique :

DÉFINITION 1.12. *Si (X, ω) est une variété symplectique, une structure presque complexe J sur X est* adaptée *à ω si l'application $(u, v) \longmapsto \omega(u, Jv)$ est une métrique Riemannienne sur X.*

Les structures adaptées forment un bon espace de paramètres, qui est non vide et contractile [**Grom**]. Les objets géométriques associés à ces structures sont les courbes pseudo-holomorphes :

Définition 1.13. *Une* courbe pseudo-holomorphe *sur* (X, ω, J)*, où* J *est adaptée à* ω*, est la donnée d'une surface de Riemann* Σ *et d'une application* $f : \Sigma \longrightarrow (X, J)$ *dont la différentielle commute avec les structures complexes sur* $T\Sigma$ *et* TX*. La première de ces structures est donnée par la structure intégrable de* Σ *et la seconde par* J*.*

Les courbes pseudo-holomorphes sont un outil très efficace pour l'étude de la topologie des variétés symplectiques (voir [**McD-Sa**]). Le non-squeezing théorème de Gromov en est un exemple :

Théorème 1.14. *Si* $u : B^{2n}(r) \longrightarrow \mathbb{R}^{2n}$ *est un plongement symplectique d'une boule dans* $\mathbb{R}^{2n}$ *tel que* $u\big(B^{2n}(r)\big) \subseteq B^2(R) \times \mathbb{R}^{2n-2}$*, alors* $r \leq R$*.*

Les courbes pseudo-holomorphes permettent aussi de définir des invariants symplectiques qui sont les invariants de Gromov-Witten. Ce sont des invariants énumératifs représentant le nombre de courbes pseudo-holomorphes de genre et de classe d'homologie fixés intersectant des cycles prescrits. Ces invariants ont été initialement considérés par Konsevitch et Manin [**Ko-Ma**] (voir aussi [**Fu-Pa**] et [**Ko-Va**]) dans le cadre de la géométrie algébrique pour résoudre le problème suivant :

Calculer le nombre N_d *de courbes rationnelles de degré* d *passant par* $3d-1$ *points génériques du plan.*

En géométrie symplectique, ces invariants ont été introduits par Tian et Ruan [**Ru-Ti**] dans le cas des variétés symplectiques semi-positives. L'un des principaux intérêts de ces invariants est de déformer l'anneau de cohomologie d'une variété symplectique.

De manière plus précise, soit (X, ω) une variété symplectique et J une structure presque complexe régulière adaptée à ω. Les invariants de Gromov-Witten les plus simples sont définis comme suit :

Définition 1.15. *Soit* $A \in H_2(X, \mathbb{Z})$ *et* α, β, $\gamma \in H^*(X, \mathbb{Z})/_{\text{torsion}}$ *des classes d'homologie duales à des sous-variétés* Z_1*,* Z_2*,* Z_3 *de* X*. L'invariant de Gromov-Witten* $GW^X_{0,A,3}(\alpha, \beta, \gamma)$ *est le nombre de courbes pseudo-homorphes rationnelles* $u : \mathbb{P}^1 \longrightarrow (X, J)$ *telles que* $u_*\big[\mathbb{P}^1\big] = A$ *et telles que, pour un choix générique de* Z_1*,* Z_2*,* Z_3*, on ait* $u(0) \in Z_1$*,* $u(1) \in Z_2$ *et* $u(\infty) \in Z_3$*, chaque courbe étant comptée avec un signe spécifique (voir* [**McD-Sa**]*).*

L'hypothèse de régularité de J est une condition de transversalité. Les structures presque complexes adaptées régulières forment une partie dense et connexe de l'ensemble des structures presque complexes adaptées [**McD-Sa**].

Si A est la classe d'homologie d'un point dans X,

$$GW^X_{0,\mathrm{pt},3}(\alpha, \beta, \gamma) = \int_X \alpha \wedge \beta \wedge \gamma = \langle \alpha \cup \beta, \gamma \rangle.$$

On voit ainsi que les invariants de Gromov-Witten pour la classe d'un point combinés avec la dualité de Poincaré décrivent le cup-produit dans $H^*(X, \mathbb{Z})/_{\text{torsion}}$.

Si $\big(e_i\big)_{1 \leq i \leq N}$ est une base de $H^*(X, \mathbb{Z})/_{\text{torsion}}$ et $g_{ij} = \displaystyle\int_X e_i \wedge e_j$, on définit les constantes de structure déformées $f^k_{ij} = \displaystyle\sum_{l=1}^{N} g^{lk} \sum_{A \in H_2(X,\mathbb{Z})} GW^X_{0,A,3}\big(e_i, e_j, e_k\big) q^{\omega(A)}$, où $\omega(A) = \displaystyle\int_A \omega$ et q est un paramètre formel. Ces constantes permettent alors de définir le *petit produit quantique* sur $H^*(X,\mathbb{Z})/_{\text{torsion}}$ par la formule

$$\alpha \times_Q \beta = \sum_k \Big(\sum_{i,j} f_{ij}^k \, \alpha^i \, \beta^j\Big) e_k.$$

Ce produit est associatif et il coïncide pour $q = 0$ avec le produit usuel [**McD-Sa**]. On peut définir de façon plus générale un *gros produit quantique*, les paramètres formels étant au nombre de r où $r = \operatorname{rang}\big(H_2(X,\mathbb{Z})\big)$ [**Fu-Pa**], [**Ko-Va**], [**McD-Sa**].

La cohomologie quantique se développe également pour les invariants de Gromov-Witten en genre supérieur [**Ru-Ti**], mais les constructions sont beaucoup plus délicates et nécessitent de perturber les courbes pseudo-holomorphes.

La théorie de Gromov-Witten a ensuite été adaptée aux orbifolds. La première étape déterminante est la construction par Chen et Ruan de l'anneau de cohomologie orbifold d'un orbifold presque complexe. L'anneau $H^*_{CR}(X,\mathbb{Q})$ est en général différent de $H^*(X,\mathbb{Q})$ et fait intervenir la cohomologie des secteurs tordus de l'orbifold [**Ch-Ru 1**], [**Ad-Le-Ru**], [**Fa-Gö**]. Les invariants de Gromov-Witten ont ensuite été définis par Abramovitch-Graber-Vistoli [**Ab-Gr-Vi**] et Chen-Ruan [**Ch-Ru 2**]. L'anneau de cohomologie orbifold se déforme comme précédemment en un anneau de cohomologie orbifold quantique.

Une des grandes conjectures actuelles reliant géométrie algébrique et symplectique est la conjecture de la résolution crêpante de Ruan :

Conjecture 1.1. [**Co-Ru**] *Soient X une variété projective complexe Gorenstein à singularités quotient et $\pi\colon \widetilde{X} \longrightarrow X$ une* résolution crêpante *de X, ce qui signifie que $\widetilde{X}$ est lisse, que π est un morphisme birationnel et que $\pi^* K_X = K_{\widetilde{X}}$. Alors les grosses algèbres de cohomologie quantique de X et $\widetilde{X}$ sont isomorphes après spécialisation des paramètres quantiques à des valeurs spécifiques.*

Pour des développements récents sur cette conjecture, voir [**Co-Co-Ir-Ts**], [**Co-Ir-Ts**] et [**Co**]. La conjecture de Ruan a été formulée et beaucoup étudiée sur des variétés algébriques, comme par exemple les espaces projectifs à poids [**Co-Co-Le-Ts**], [**Bo-Ma-Pe 1**], [**Bo-Ma-Pe 2**] (les singularités sont en effet difficiles à traiter en dehors de ce cadre). Un cas particulièrement fécond est celui des produits symétriques d'une surface projective lisse.

Si X est une surface projective complexe lisse et n un entier positif, le schéma de Hilbert ponctuel $X^{[n]}$ est l'ensemble des 0-cycles de X de longueur n. Un point ξ de $X^{[n]}$ est donc un faisceau d'idéaux $\mathcal{J}$ de $\mathcal{O}_X$ tel que :

– $\mathcal{O}_X/_{\mathcal{J}}$ est supporté en un nombre fini de points $x_1, \ldots, x_k$. Ces points constituent le support de ξ.

– $\displaystyle\sum_{i=1}^k \ell_{x_i}(\xi) = n$, où $\ell_{x_i}(\xi) = \dim_{\mathbb{C}}\big(\mathcal{O}_X/_{\mathcal{J}}\big)_{|x_i}$ est la longueur de ξ en x_i.

Le schéma de Hilbert $X^{[n]}$ est une variété projective.

Le morphisme de Hilbert-Chow $HC\colon X^{[n]} \longrightarrow S^n X$ est défini par

$$HC(\xi) = \sum_{x \in \operatorname{supp}(\xi)} \ell_x(\xi) x.$$

Ce morphisme est bijectif au dessus de la strate dense de S^nX formée des n-listes de points deux à deux distincts. Les deux théorèmes fondamentaux sur les schémas de Hilbert ponctuels d'une surface projective lisse sont ceux de Fogarty et Briançon :

Théorème 1.16. [**Fo**] *Si X est une surface projective lisse, $X^{[n]}$ est lisse de dimension $2n$.*

Théorème 1.17. [**Br**] *Si X est une surface projective lisse, pour tout x de X, $HC^{-1}(nx)$ est une variété algébrique irréductible de dimension $n-1$.*

Dans le cas $n = 2$, $X^{[2]}$ admet une description particulièrement simple : c'est le quotient de l'éclaté de $X \times X$ le long de la diagonale par l'action naturelle de $\mathbb{Z}/_{2\mathbb{Z}}$ qui permute les facteurs ; la fibre maximale du morphisme de Hilbert-Chow est alors isomorphe à $\mathbb{P}^1$. Si $n \geq 3$, cette fibre maximale est irréductible par le théorème de Briançon, mais elle n'est plus lisse.

Le produit symétrique S^nX est une variété projective singulière Gorenstein. Le théorème de Fogarty montre que $X^{[n]}$ est une résolution des singularités de S^nX. Cette résolution est de plus crêpante.

Bien avant que ne soit formulée la conjecture de la résolution crêpante, les nombres de Betti des schémas de Hilbert ponctuels ont été calculés par Göttsche (pour l'historique de cette formule, voir [**Na 2**]) :

Théorème 1.18. [**Gö**], [**Gö-So**] *Si X est une surface quasi-projective lisse, et si $n \in \mathbb{N}^*$,*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{4n} b_i\big(X^{[n]}\big) t^i q^n = \prod_{m=1}^{\infty} \frac{\big[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\big]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

Ce résultat se déduit également d'un résultat de Batyrev [**Ba**] qui démontre une version additive de la conjecture de la résolution crêpante.

Après la contribution de Göttsche, les travaux de Nakajima [**Na 1**], puis ceux de Lehn [**Le**], Li, Qin et Wang [**Li-Qi-Wa 2**], Lehn et Sorger [**Le-So 1**], Fantechi et Göttsche [**Fa-Gö**], ont permis une étude approfondie de la structure multiplicative des anneaux de cohomologie $H^*\big(X^{[n]}, \mathbb{Q}\big)$, dont un aboutissement a été la preuve de la conjecture de la résolution crêpante pour les schémas de Hilbert ponctuels par Lehn et Sorger dans le cas où X est une surface $K3$.

Théorème 1.19. [**Le-So 2**] *Si X est une surface $K3$, pour tout $n \in \mathbb{N}^*$, on a un isomorphisme d'anneaux gradués*

$$H^*\big(X^{[n]}, \mathbb{Q}\big) \simeq H^*_{CR}\big(S^nX, \mathbb{Q}\big).$$

L'étude des schémas de Hilbert s'est poursuivie au delà de la cohomologie. Ellinsgrud, Göttsche et Lehn ont fourni une contribution importante en étudiant la classe de $TX^{[n]}$ en K-théorie.

Théorème 1.20. [**El-Gö-Le**] *Soit X une surface projective lisse. Les nombres de Chern de $X^{[n]}$ se calculent par des formules universelles à partir des nombres de Chern de X.*

Ce théorème est équivalent à la propriété remarquable suivante :

La classe de cobordisme algébrique de $X^{[n]}$ ne dépend que de la classe de cobordisme algébrique de X.

On voit donc que le schéma de Hilbert ponctuel a fourni un extraordinaire exemple pour l'étude la conjecture de la résolution crêpante. Cependant, l'hypothèse algébrique n'est probablement pas déterminante, et cette conjecture semble plutôt de nature symplectique.

Des exemples explicites de résolutions crêpantes dans le cadre presque complexe et symplectique sont fournis par les schémas de Hilbert ponctuels $X^{[n]}$ d'une variété symplectique ou presque complexe X de dimension 4 qui sont des désingularisations des produits symétriques S^nX. Ils ont été construits par Voisin dans [**Vo 1**] et [**Vo 2**].

L'objet de notre travail dans le second article est d'étudier ces schémas de Hilbert presque complexes et symplectiques. Pour adapter à notre cadre la théorie classique sur les surfaces projectives, nous utilisons, au lieu d'une structure intégrable, les structures intégrables relatives introduites dans la construction de Voisin.

DÉFINITION 1.21. *Soit* (X, J) *une variété presque complexe de dimension* 4. *Une* structure relative intégrable *paramétrée par* S^nX *est une famille de structures intégrables* $\big(J_{\underline{x}}\big)_{\underline{x}\in S^nX}$ *variant de manière* $\mathcal{C}^\infty$ *avec* $\underline{x}$ *telle que, pour tout* $\underline{x} \in S^nX$, $J_{\underline{x}}$ *soit une structure intégrable sur un ouvert* $W_{\underline{x}}$ *de* X *contenant les points de* $\underline{x}$.

De telles structures existent et forment, si elles sont choisies suffisamment proches de J, un ensemble contractile. Réciproquement, toute structure relative intégrable paramétrée par un produit symétrique S^nX définit une structure presque complexe sur X. Si J est une telle structure, le *schéma de Hilbert presque complexe de Voisin* est alors défini de façon ensembliste par

$$X^{[n]} = \bigsqcup_{\underline{x}\in S^nX} HC^{-1}_{W_{\underline{x}},\, J_{\underline{x}}}(\,\underline{x}\,)$$

où $HC_{W_{\underline{x}},\, J_{\underline{x}}} : W_{\underline{x}}^{[n]} \longrightarrow S^nW_{\underline{x}}$ est le morphisme de Hilbert-Chow associé à la structure intégrable $J_{\underline{x}}$ sur $W_{\underline{x}}$.

Nous allons maintenant citer les principaux résultats obtenus. Les premiers concernent seulement la structure additive de l'anneau de cohomologie des schémas de Hilbert presque complexes et correspondent dans le cas des surface projectives lisses aux travaux de Göttsche et Nakajima [**Gö**], [**Gö-So**], [**Na 1**].

THÉORÈME 1.22. *Soit* (X, J) *une surface presque complexe compacte de dimension* 4.

(i) *Soit* $\Big(b_i\big(X^{[n]}\big)\Big)_{i=0\ldots 4n}$ *la suite des nombres de Betti du schéma de Hilbert presque complexe* $X^{[n]}$. *La série génératrice de ces nombres de Betti est donnée par*

$$\sum_{n=0}^{\infty}\sum_{i=0}^{4n} b_i\big(X^{[n]}\big)t^iq^n = \prod_{m=1}^{\infty} \frac{\big[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\big]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

(ii) *On peut construire des opérateurs de Nakajima* $q_i(\alpha)$, $i \in \mathbb{Z}$, $\alpha \in H^*(X, \mathbb{Q})$ *tels que :*

– *les relations de Nakajima suivantes sont vérifiées :*

$$\forall\, i, j \in \mathbb{Z}, \quad \forall\, \alpha, \beta \in H^*(X, \mathbb{Q}) \quad \big[q_i(\alpha),\, q_j(\beta)\big] = i\,\delta_{i+j,0}\int_X \alpha\beta,$$

– *si* $\mathbb{H} = \bigoplus_{n=0}^{\infty} H^*\big(X^{[n]}, \mathbb{Q}\big)$, *les opérateurs de Nakajima définissent une représentation irréductible de l'algèbre de Heisenberg* $\mathcal{H}\big(H^*(X, \mathbb{Q})\big)$ *dans* $\mathbb{H}$ *avec* 1 *comme vecteur de plus haut poids.*

Les opérateurs de Nakajima étant construits, on adapte la théorie de Lehn et Li-Qin-Wang pour obtenir une description de la structure multiplicative de $H^*(X^{[n]}, \mathbb{Q})$. La principale étape est le calcul de l'opérateur de bord ∂ défini par Lehn dans [**Le**].

Définition 1.23. *Soit (X,J) une variété presque complexe et $X^{[n]}$ le schéma de Hilbert presque complexe associé. L'opérateur de bord ∂ est l'endomorphisme de $\mathbb{H}$ défini par*

$$\partial(\alpha)=-\frac{1}{2}\big[\partial X^{[n]}\big]\cup\alpha,$$

où $\partial X^{[n]}$ est l'ensemble des schémas qui ne sont pas supportés en n points distincts et $\big[\partial X^{[n]}\big]$ est la classe de cycle associée dans $H^2\big(X^{[n]},\mathbb{Q}\big)$.

Nous obtenons le résultat suivant, dû à Lehn [**Le**] dans le cas intégrable :

Théorème 1.24. *Soit (X,ω) une variété symplectique compacte de dimension 4 et J une structure presque complexe adaptée à ω. Si ∂ est l'opérateur de bord, alors pour tous $n,m\in\mathbb{Z}$ et tous $\alpha,\beta\in H^*(X,\mathbb{Q})$, on a l'identité :*

$$\big[\partial\, q_n(\alpha)-q_n(\alpha)\partial, q_m(\beta)\big]=-nm\left\{q_{n+m}(\alpha\beta)-\frac{|n|-1}{2}\,\delta_{n+m,0}\left(\int_X c_1(X)\alpha\beta\right)\mathrm{id}\right\}.$$

Nous supposons que X est symplectique pour pouvoir utiliser les résultats de Donaldson [**Do**]. Des arguments formels dus à Li-Qin-Wang [**Li-Qi-Wa 1**], [**Li-Qi-Wa 3**], [**Li-Qi-Wa 2**] et Lehn-Sorger [**Le-So 1**], [**Le-So 2**] nous permettent d'obtenir des conséquences importantes du théorème précédent :

Théorème 1.25. *Soit (X,ω) une variété symplectique compacte de dimension 4. On suppose que $b_1(X)=0$.*

(i) *L'anneau de cohomologie $H^*\big(X^{[n]},\mathbb{Q}\big)$ du schéma de Hilbert symplectique $X^{[n]}$ peut être construit par des formules universelles à partir de $H^*(X,\mathbb{Q})$, $c_1(X)$ et $c_2(X)$.*

(ii) *Si la première classe de Chern de X est nulle dans $H^2(X,\mathbb{Q})$, $H^*\big(X^{[n]},\mathbb{Q}\big)$ est isomorphe à l'anneau de cohomologie orbifold $H^*_{CR}\big(S^nX,\mathbb{Q}\big)$ du produit symétrique S^nX.*

Dans le cas algébrique, la première partie du théorème est due à Li, Qin et Wang [**Li-Qi-Wa 3**].

La conjecture de la résolution crêpante de Ruan est ainsi vérifiée pour les produits symétriques de surfaces symplectiques simplement connexes à première classe de Chern nulle. Ceci étend le résultat obtenu par Lehn et Sorger pour les surfaces $K3$ [**Le-So 2**].

Tous ces énoncés ne font intervenir que l'espace topologique sous-jacent à $X^{[n]}$. La construction de Voisin munit $X^{[n]}$ de structures plus riches que celle d'un espace topologique : $X^{[n]}$ est une variété différentiable munie d'une structure presque complexe stable et $X^{[n]}$ est symplectique si X est symplectique [**Vo 2**]. Le fibré tangent $TX^{[n]}$ définit alors un élément κ_n dans l'anneau de K-théorie complexe de $X^{[n]}$ et la classe de cobordisme presque complexe de $X^{[n]}$ définie par la structure stable précédente est caractérisée par les nombres de Chern de κ_n.

Les nombres de Chern de $X^{[n]}$ ont été étudiés dans le cas projectif par Ellinsgrud, Göttsche et Lehn dans [**El-Gö-Le**], comme on l'a vu plus haut. Ils s'obtiennent de manière universelle à partir des nombres de Chern de X. La preuve de ce résultat utilise intensivement la K-théorie des faisceaux cohérents. Nous généralisons cette démonstration au cadre presque complexe en considérant des faisceaux cohérents relatifs sur des espaces topologiques fibrés en ensembles analytiques associés à des structures relatives intégrables. Nous obtenons le

Théorème 1.26. *Soit (X,J) une variété presque complexe compacte de dimension 4 et $X^{[n]}$ le schéma de Hilbert presque complexe associé. Si P est un polynôme pondéré dans $\mathbb{Q}\big[T_1,\dots,T_{2n}\big]$*

de degré $4n$ *où* T_i *est de degré* $2i$*, il existe alors un polynôme pondéré* $\widetilde{P} \in \mathbb{Q}\big[T_1, T_2\big]$ *de degré* 4 *dépendant seulement de* P *et de* n *tel que*

$$\int_{X^{[n]}} P\big(c_1(X^{[n]}), \ldots, c_{2n}(X^{[n]})\big) = \int_X \widetilde{P}\big(c_1(X), c_2(X)\big).$$

Ce théorème signifie que la classe de cobordisme de $X^{[n]}$ ne dépend que de la classe de cobordisme de X.

Les structures relatives de Voisin permettent donc d'adapter la théorie classique aux variétés presque complexes ou symplectiques, ce qui montre a posteriori que c'est bien dans ce cadre qu'il convient de considérer les schémas de Hilbert ponctuels. Un prolongement de ce travail pourrait être le calcul des invariants de Gromov-Witten pour les schémas de Hilbert ponctuels symplectiques ; ce problème est posé dans [**Vo 2**].

2. English version

The thesis we present here consists of two articles: "Chern classes in Deligne cohomology for coherent analytic sheaves" and "Topological properties of punctual Hilbert schemes of symplectic 4-dimensional manifolds". These two topics belong indeed to the same field of investigation, that is to construct or to adapt some objects coming from algebraic geometry in other settings: non projective complex manifolds in the first case, almost-complex and symplectic geometry in the second one.

2.1. Chern classes in Deligne cohomology for coherent analytic sheaves.

In the first article, we concern ourselves with coherent analytic sheaves and their possibly associated Chern classes. The theory of coherent sheaves developed at the same time both in the algebraic and in the analytic contexts [**Se 1**], [**Ca**], [**Se 2**], [**Ca-Se**], [**Gr-Re**], [**Gr-Ri**]. On a projective complex manifold, according to Serre GAGA principle [**Se 2**], the category of coherent algebraic sheaves is equivalent to the category of coherent analytic sheaves, so that there is no point in differentiating the two theories. In a more general case, some important results can be formulated in the same terms for coherent algebraic or analytic sheaves. The proofs are nevertheless often very different. Let us mention three examples.

The first one is the finiteness theorem, namely:

Theorem 2.1. *Let X be a complex projective (resp. complex compact) manifold and $\mathcal{F}$ a coherent algebraic (resp. analytic) sheaf on X. Then the $H^i(X,\mathcal{F})$ are finite dimensional vector spaces.*

The proof for coherent algebraic sheaves is reduced by dévissage to the special case where $X=\mathbb{P}^n$ and $\mathcal{F}=\mathcal{O}(k)$. For coherent analytic sheaves on a complex manifold, the proof, which is due to Cartan and Serre [**Ca-Se**], [**Gr-Re**], uses Montel's theorem and functional analysis methods.

This theorem admits a relative version:

Theorem 2.2. *Let X and Y be algebraic (resp. complex) manifolds, $f\colon X\longrightarrow Y$ an algebraic (resp. holomorphic) morphism, and $\mathcal{F}$ a coherent algebraic (resp. analytic) sheaf on X. If f is proper on the support of $\mathcal{F}$, then $f_*\mathcal{F}$ as well as the higher direct images $R^if_*\mathcal{F}$ are coherent algebraic (resp. analytic) sheaves on Y.*

In the algebraic setting the result is obtained by dévissage again [**Bo-Se**]. The proof by Grauert and Riemenschneider [**Gr-Ri**] for complex manifolds is much more difficult and uses finiteness theorems for nuclear Fréchet spaces.

We can take as a third example one of the most important results in the theory of coherent algebraic sheaves, namely the Grothendieck-Riemann-Roch theorem.

Theorem 2.3. *Let X and Y be smooth quasi-projective manifolds, $f\colon X\longrightarrow Y$ a proper map and $\mathcal{F}$ a coherent algebraic sheaf on X. Then the following identity holds in the Chow ring of Y*

$$f_*\big(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(X)\big)=\sum_{i\geq 0}(-1)^i\,\mathrm{ch}\big(R^if_*(\mathcal{F})\big)\,\mathrm{td}(Y).$$

If we take for X a projective manifold and for Y the manifold reduced to a point, we obtain the Riemann-Roch-Hirzebruch formula, which was established by cobordism methods in the first place [**Hirz**]:

$$\chi(X,\mathcal{F})=\int_X \mathrm{ch}(\mathcal{F})\,\mathrm{td}(X).$$

If $\mathcal{E}$ is a locally free sheaf on a complex compact manifold X, the Riemann-Roch-Hirzebruch formula still holds for $\mathcal{E}$. It is a consequence of Atiyah-Singer's formula, which computes the index of an elliptic operator (in this case the $\overline{\partial}$ operator). More generally, Toledo and Tong have proved this formula for an arbitrary coherent analytic sheaf by constructing an explicit parametrix for the $\overline{\partial}$ operator [**To-To 1**].

When X and Y are complex compact manifolds and the Chern classes are chosen in the Hodge rings $\bigoplus\limits_i H^i\big(X, \Omega^i_X\big)$ and $\bigoplus\limits_i H^i\big(Y, \Omega^i_Y\big)$ of X and Y, the Grothendieck-Riemann-Roch equality still holds. The proof was carried out by O'Brian, Toledo and Tong [**OB-To-To**]. There are nevertheless still some cases where the Grothendieck-Riemann-Roch theorem remains unproved, for instance when the manifolds are non Kähler and the Chern classes are chosen in $H^*(X, \mathbb{Q})$ and $H^*(Y, \mathbb{Q})$.

It is interesting to compare the Chern classes theory in the algebraic and analytic settings. For locally free sheaves, by Grothendieck's splitting principle [**Grot**], [**Vo 3**], these classes can be constructed in both contexts. The chosen cohomology rings on algebraic manifolds are the Chow rings, and they are the most general possible, whereas on non projective complex manifolds, the best cohomology theory which is known is the Deligne-Beilinson cohomology [**Es-Vi**], [**Vo 3**].

The theory becomes quite different for coherent sheaves. On algebraic manifolds, the construction of Chern classes is complete: it is done in the Chow rings and, via the existence of global locally free resolutions, reduces to the theory for locally free sheaves. The following counterexample of Voisin forbids to transpose this method to abstract complex manifolds.

THEOREM 2.4. [**Vo 4**] *On a generic complex torus of dimension at least* 3*, the ideal sheaf of a point does not admit a global locally free resolution.*

Before our work, several Chern classes theories for coherent analytic sheaves were already existing in different specific cohomology rings: the Atiyah-Hirzebruch classes in $H^*(X, \mathbb{Z})$ [**At-Hi**], the Atiyah classes $\bigoplus\limits_i H^i\big(X, \Omega^i_X\big)$ [**At**] and the Green ones in $\bigoplus\limits_i \mathbb{H}^{2i}\big(X, \Omega^{\bullet \geqslant i}_X\big)$ [**Gre**], [**To-To 2**].

Our aim in "Chern classes in Deligne cohomology for coherent analytic sheaves" is to construct unified Chern classes for coherent analytic sheaves on a complex compact manifold, with values in the rational Deligne cohomology of the base. To do so, we develop a new approach.

Voisin's counterexample [**Vo 4**] shows that complex manifolds possess so few holomorphic vector bundles that it is not possible to think of studying an arbitrary coherent sheaf using such vector bundles.

Among coherent sheaves we can consider on the one hand locally free sheaves and on the other hand torsion sheaves; the latter are coherent sheaves whose supports are proper analytic subsets of the base X. If Z is a smooth submanifold of X and $\mathcal{F}$ a torsion sheaf supported in Z of the type $i_{Z*}\mathcal{G}$ where $\mathcal{G}$ is coherent on Z, the Grothendieck-Riemann-Roch formula for the injection $i_Z : Z \longrightarrow X$ predicts the equality:

$$\mathrm{ch}(\mathcal{F}) = i_{Z*}\big(\mathrm{ch}(\mathcal{G})\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\big)$$

and thus allows us *a priori* to define $\mathrm{ch}(\mathcal{F})$ by this formula if $\mathrm{ch}(\mathcal{G})$ is known. This observation leads us to perform the construction of the Chern classes by induction on the dimension of X.

So doing, we are able to define Chern classes for a specific class of torsion sheaves. A first problem arises, for it is not clear whether $\mathrm{ch}(Z)$ is independent of Z. On the other hand, the support of a torsion sheaf is not a priori included in a smooth submanifold of X. It follows that the construction of the Chern classes of a general torsion sheaf cannot be performed directly on X. At this point, it appears necessary to change the basis and to consider bimeromorphic models of X. Hironaka's desingularization theorem then enables us to solve these problems and to build a coherent theory of Chern classes for torsion sheaves.

Now if $\mathcal{F}$ is any coherent sheaf and if $\mathcal{F}_{\text{tor}} \subseteq \mathcal{F}$ is the subsheaf of the torsion elements, $\mathcal{F}$ will be said to be *locally free modulo torsion* if $\mathcal{F}/_{\mathcal{F}_{\text{tor}}}$ is locally free. If $\mathcal{F}$ is locally free modulo torsion, the exact sequence

$$0 \longrightarrow \mathcal{F}_{\text{tor}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/_{\mathcal{F}_{\text{tor}}} \longrightarrow 0$$

leads us to define $\operatorname{ch}(\mathcal{F})$ by the relation

$$\operatorname{ch}(\mathcal{F}) = \operatorname{ch}\big(\mathcal{F}_{\text{tor}}\big) + \operatorname{ch}\Big[\mathcal{F}/_{\mathcal{F}_{\text{tor}}}\Big].$$

Nevertheless, a coherent sheaf is not always locally free modulo torsion. For instance, if Z is an analytic subset of X of codimension at least 2, the ideal sheaf $\mathcal{J}_Z$ is torsion-free but not locally free. To define the Chern classes of a coherent sheaf of positive generic rank, we use, and this is a key argument in the construction, the following desingularization theorem:

THEOREM 2.5. [**Ro**] *Let $\mathcal{F}$ be a coherent sheaf on X of generic rank r. There exists a bimeromorphic morphism $\pi\colon \widetilde{X} \longrightarrow X$ obtained by a finite sequence of blowups with smooth centers, and a locally free sheaf $\mathcal{Q}$ on $\widetilde{X}$ of rank r, such that there exists a surjective morphism $\pi^*\mathcal{F} \longrightarrow \mathcal{Q}$.*

This theorem can be intuitively formulated in the following way:

If $\mathcal{F}$ is a coherent sheaf, it is possible to transform by a succession of blowups the whole singularity of $\mathcal{F}$ into torsion.

This result is an immediate consequence of Hironaka's flattening theorem [**Hiro**]. The initial proof in [**Ro**] relies only on Hironaka's desingularization theorem.

Using the arguments above, we are able to define $\operatorname{ch}(\mathcal{F})$ for any coherent sheaf $\mathcal{F}$ on a compact manifold of dimension n when the theory is known for manifolds of dimension at most $n-1$. The so constructed classes satisfy by definition the Grothendieck-Riemann-Roch formula for the immersion of a smooth divisor.

The induction compels us to verify for the so constructed classes some properties such as functoriality under pullbacks, the product formula and the Whitney additivity formula. The latter means that for any exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$$

of coherent sheaves, $\operatorname{ch}(\mathcal{G}) = \operatorname{ch}(\mathcal{F}) + \operatorname{ch}(\mathcal{H})$; it is not a formal consequence of the construction. Indeed, if X is projective and if $\mathcal{F}$ is a coherent sheaf on X of the type $i_{Z*}\mathcal{G}$, where Z is a smooth hypersurface of X, our construction prescribes the Grothendieck-Riemann-Roch formula while defining $\operatorname{ch}\big(i_{Z*}\mathcal{G}\big)$. On the other hand, $\mathcal{F}$ admits a global locally free resolution. If the Whitney formula holds, it means that the Grothendieck-Riemann-Roch formula has been established for $i_{Z*}\mathcal{G}$, which is by no means obvious.

Finally, we obtain the following result:

THEOREM 2.6. *Let X be a complex compact manifold. For any coherent sheaf on X, it is possible to define a Chern character $\operatorname{ch}(\mathcal{F})$ in $\oplus_p H^{2p}_{\text{Del}}(X, \mathbb{Q}(p))$ such that:*

(i) *for any exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ of coherent sheaves on X, we have $\operatorname{ch}\mathcal{G} = \operatorname{ch}(\mathcal{F}) + \operatorname{ch}(\mathcal{H})$,*

(ii) *if* $f : X \longrightarrow Y$ *is holomorphic and if* y *is an element of* $K(Y)$*, then* $\mathrm{ch}(f^{\dagger}y) = f^{*}\,\mathrm{ch}(y)$*,*[2]
(iii) *if* $\mathcal{E}$ *is a locally free sheaf, then* $\mathrm{ch}(\mathcal{E})$ *is the usual Chern character in rational Deligne cohomology obtained by Grothendieck's splitting principle,*
(iv) *if* Z *is a smooth closed submanifold of* X *and if* $\mathcal{F}$ *is a coherent sheaf on* X*, then the Grothendieck-Riemann-Roch theorem holds for* $\big(i_Z, \mathcal{F}\big)$*, i.e.*

$$\mathrm{ch}\big(i_{Z*}\mathcal{F}\big) = i_{Z*}\Big(\mathrm{ch}(\mathcal{F})\ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big).$$

The Chern classes obtained in this way satisfy by their very construction the Grothendieck-Riemann-Roch theorem for the immersion of a smooth hypersurface. By blowup, we deduce the Grothendieck-Riemann-Roch theorem for the immersion of a smooth submanifold of any codimension.

The most general form of the Grothendieck-Riemann-Roch theorem we obtain is the following:

THEOREM 2.7. *The Grothendieck-Riemann-Roch theorem holds in rational Deligne cohomology for projective morphisms between compact complex manifolds.*

The result is obtained by decomposing the morphism f into an immersion $i\colon X \longrightarrow Y \times \mathbb{P}^N$ and the projection $pr_1\colon Y \times \mathbb{P}^N \longrightarrow Y$. The Grothendieck-Riemann-Roch theorem already holds for i. For the projection pr_1, using the article [**Bei**], we can reduce the proof to the case where Y is a point.

Let us remark that if the manifolds are projective, Theorem 2.7 is the classical Grothendieck-Riemann-Roch formula of [**Bo-Se**]. The arguments above hold in the Chow rings for projective manifolds and so provide a new proof of the Grothendieck-Riemann-Roch theorem in the algebraic case.

Now the next point is naturally to examine the unicity of the so constructed Chern classes. We prove a general unicity result. Let us consider a cohomology theory $X \longrightarrow A(X)$ defined on compact complex manifolds. We suppose that $A(X) = \oplus_{i=0}^{\dim X} A^i(X)$ is a graded $\mathbb{Q}$-algebra satisfying the following properties:

(α) For any holomorphic map $f\colon X \longrightarrow Y$, we are given a functorial pullback morphism $f^{*}\colon A(Y) \longrightarrow A(X)$, which is a morphism of graded algebras.
(β) If σ is a blowup of a smooth compact complex manifold along a smooth submanifold, then σ^{*} is injective.
(γ) If E is a holomorphic vector bundle on X and $\pi : \mathbb{P}(E) \longrightarrow X$ is the associated projective fiber bundle, then π^{*} is injective.
(δ) If X is a smooth compact complex manifold and if Y is a smooth submanifold of codimension d, there exists a Gysin morphism $i_{*}\colon A^{*}(Y) \longrightarrow A^{*+d}(X)$.

The unicity theorem is stated in the following way:

THEOREM 2.8. *Suppose that* ch *and* ch$'$ *are two Chern characters for coherent sheaves on smooth compact complex manifolds, with values in a cohomology theory verifying* (α)*,* (β)*,* (γ) *and* (δ)*, such that:*

(i) *the Whitney formula holds for* ch *and* ch$'$*,*
(ii) *the functoriality formula under pullbacks holds for* ch *and* ch$'$*,*

[2] For the definition of $f^{\dagger}$, see page 46.

(iii) *if L is a holomorphic line bundle, then* $\mathrm{ch}(L) = \mathrm{ch}'(L)$,
(iv) *in both theories, the Grothendieck-Riemann-Roch formula holds for immersions.*

Then for any coherent sheaf $\mathcal{F}$, $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}'(\mathcal{F})$.

For the Chern classes already known in cohomologies weaker than Deligne cohomology, the Whitney formula and the functoriality formula hold. If the Grothendieck-Riemann-Roch formula for immersions holds for these classes, they coincide with the classes we have constructed, and the converse is true.

We obtain the following comparison theorem:

THEOREM 2.9. *Let $\mathcal{F}$ be a coherent sheaf on X. Then*

(i) *The classes $c_i(\mathcal{F})$ in $H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i))$ and $c_i(\mathcal{F})^{\mathrm{top}}$ in $H^{2i}(X, \mathbb{Z})$ have the same image in $H^{2i}(X, \mathbb{Q})$.*
(ii) *The image of $c_i(\mathcal{F})$ by the natural morphism from $H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i))$ to $H^i(X, \Omega^i_X)$ is the i-th Atiyah-Chern class of $\mathcal{F}$.*

The Grothendieck-Riemann-Roch formula for immersions has been proved by Atiyah and Hirzebruch for the topological classes in [**At-Hi**] and by Toledo et Tong for the Atiyah classes in [**OB-To-To**].

Even if complexity reasons in the computations have led us to consider preferentially the exponential Chern classes, it seems possible to extend the results to the integral Deligne cohomology, and so to take account of the torsion phenomena.

One result is still missing, namely the Grothendieck-Riemann-Roch formula for any morphism between complex compact manifolds. It is certainly a difficult problem, since it is still unsolved in the particular case of the rational Betti cohomology on non Kähler manifolds (in the case of Kähler manifolds, the Chern classes in Dolbeault and rational Betti cohomology are compatible via the Hodge decomposition and the result is a consequence of [**OB-To-To**]).

2.2. Topological properties of punctual Hilbert schemes of symplectic fourfolds. Our second subject is carried out in the setting of almost-complex geometry. This geometry first appeared as an extension of the complex geometry. An almost-complex structure on a differentiable manifold M is given by an endomorphism of the tangent bundle TM whose square is $-\mathrm{id}$. It is then easier to deal with such a structure than with an integrable one, which is defined by local charts. Using the following theorem:

THEOREM 2.10. *On a two dimensional manifold, an almost-complex structure is always integrable,*

it is possible, in dimension 2, to parametrize the complex structures in an amazingly simple way. This constitutes one of the starting points of Teichmüller theory. On manifolds of even dimension larger than 2, non integrable almost-complex structures do appear. Then it is possible to give a necessary and sufficient condition for such a structure to be integrable: it is the Newlander-Nirenberg theorem.

THEOREM 2.11. *Let M be a $\mathcal{C}^\infty$ manifold endowed with an almost-complex structure J. Then J is integrable if and only if $\left[T^{1,0}M, T^{1,0}M\right] \subset T^{1,0}M$.*

The almost-complex structures form a good space of parameters in which, using this theorem, we can study the integrable ones. This approach is at the core of Kodaira's theory of deformations. In this theory, Kuranishi's theorem parametrizes by an analytic subset of the almost-complex structures the integrable ones which are close to a given integrable structure.

The almost-complex structures are used here as a space of parameters, but their own geometric properties do not interfere. The approach changes with Gromov, who does not use these structures on complex manifolds, but on symplectic ones [**Grom**].

The useful notion in this study is the almost-complex structure adapted to a symplectic form:

DEFINITION 2.12. If (X, ω) is a symplectic manifold, an almost-complex structure is *adapted* to ω if the map $(u, v) \longmapsto \omega(u, Jv)$ is a Riemannian metric on X.

The adapted structures form a good space of parameters, which is non-empty and contractible [**Grom**]. The geometrical objects adapted to these structures are the pseudo-holomorphic curves.

DEFINITION 2.13. A *pseudo-holomorphic curve* on (X, ω, J), where J is adapted to ω, is a pair consisting of a Riemann surface Σ and a map $f : \Sigma \longrightarrow (X, J)$ whose differential commutes with the complex structures on $T\Sigma$ and TX. The first of these structures is given by the integrable structure on Σ and the second by J.

The pseudo-holomorphic curves are a very useful tool to study the topology of symplectic manifolds (see [**McD-Sa**]). The nonsqueezing theorem of Gromov is an example of this:

THEOREM 2.14. *If $u : B^{2n}(r) \longrightarrow \mathbb{R}^{2n}$ is a symplectic imbedding of a ball into $\mathbb{R}^n$ such that $u\big(B^{2n}(r)\big) \subseteq B^2(R) \times \mathbb{R}^{2n-2}$, then $r \leq R$.*

The pseudo-holomorphic curves make it possible to define symplectic invariants, namely the Gromov-Witten invariants. They are enumerative invariants which represent the number of holomorphic curves of fixed genus and homology class intersecting prescribed cycles. These invariants were first considered by Konsevitch and Manin [**Ko-Ma**] (see also [**Fu-Pa**] and [**Ko-Va**]) in the context of algebraic geometry to solve the following problem:

Compute the number N_d of rational plane curves of degree d passing through $3d-1$ generic points.

In the setting of symplectic geometry, these invariants have been introduced by Tian and Ruan [**Ru-Ti**] for semi-positive symplectic manifolds. One of the main interests of these invariants is to deform the cohomology ring of a symplectic manifold.

More precisely, if (X, ω) is a symplectic manifold and J a regular almost-complex structure adapted to ω, the simplest Gromov-Witten invariants are defined in the following way:

DEFINITION 2.15. Let $A \in H_2(X, \mathbb{Z})$ and α, β, $\gamma \in H^*(X, \mathbb{Z})/_{\text{torsion}}$ be homology classes dual to submanifolds Z_1, Z_2, Z_3 of X. The Gromov-Witten invariant $GW^X_{0,A,3}(\alpha, \beta, \gamma)$ is the number of pseudo-holomorphic rational curves $u : \mathbb{P}^1 \longrightarrow (X, J)$ such that $u_*\big[\mathbb{P}^1\big] = A$ and that, for a generic choice of Z_1, Z_2, Z_3, $u(0) \in Z_1$, $u(1) \in Z_2$ and $u(\infty) \in Z_3$, each curve being counted with a specific sign (see [**McD-Sa**]).

The assumption that J is regular is a transversality condition. Regular adapted almost-complex structures are dense and connected in the set of all the adapted almost-complex structures [**McD-Sa**].

If A is the homology class of a point in X,

$$GW^X_{0,\text{pt},3}(\alpha, \beta, \gamma) = \int_X \alpha \wedge \beta \wedge \gamma = \langle \alpha \cup \beta, \gamma \rangle.$$

So we can see that the Gromov-Witten invariants for the class of a point together with the Poincaré duality describe the cup-product in $H^*(X,\mathbb{Z})/_{\text{torsion}}$.

If $\left(e_i\right)_{1\leq i\leq N}$ is a basis of $H^*(X,\mathbb{Z})/_{\text{torsion}}$ and $g_{ij}=\displaystyle\int_X e_i\wedge e_j$, the deformed structure constants are defined by $f_{ij}^k=\displaystyle\sum_{l=1}^{N} g^{lk}\sum_{A\in H_2(X,\mathbb{Z})} GW_{0,A,3}^X\big(e_i,e_j,e_k\big)q^{\omega(A)}$, where $\omega(A)=\displaystyle\int_A\omega$ and q is a formal parameter. With these constants we can define the *small quantum product* on $H^*(X,\mathbb{Z})/_{\text{torsion}}$ by the formula

$$\alpha\times_Q\beta=\sum_k\Big(\sum_{i,j} f_{ij}^k\,\alpha^i\,\beta^j\Big)e_k.$$

This product is associative and coincides for $q=0$ with the usual product [**McD-Sa**]. More generally we can define a *big quantum product*, the number of formal parameters being equal to r where $r=\operatorname{rang}\big(H_2(X,\mathbb{Z})\big)$ [**Fu-Pa**], [**Ko-Va**], [**McD-Sa**].

Quantum cohomology can be developed too for Gromov-Witten invariants of superior genus [**Ru-Ti**], but the constructions are much more difficult and one needs to perturb the pseudo-holomorphic curves.

The Gromov-Witten theory has been adapted to orbifolds. The first decisive step is the construction by Chen and Ruan of the orbifold cohomology ring of an almost complex orbifold. The ring $H^*_{CR}(X,\mathbb{Q})$ is generally different from $H^*(X,\mathbb{Q})$ and takes into account the cohomology of the twisted sectors of the orbifold [**Ch-Ru 1**], [**Ad-Le-Ru**], [**Fa-Gö**]. The Gromov-Witten invariants have been defined by Abramovitch-Graber-Vistoli [**Ab-Gr-Vi**] and Chen-Ruan [**Ch-Ru 2**]. The orbifold cohomology ring can be deformed as before into a quantum orbifold cohomology ring.

One of the great conjectures connecting algebraic and symplectic geometry is Ruan's crepant resolution conjecture :

Conjecture 2.16. [**Co-Ru**] *Let X be a projective complex Gorenstein manifold with quotient singularities and let $\pi\colon\widetilde{X}\longrightarrow X$ be a crepant resolution of X, which means that $\widetilde{X}$ is smooth, π is a birational morphism and $\pi^*K_X=K_{\widetilde{X}}$. Then the big quantum cohomology algebras of X and $\widetilde{X}$ are isomorphic after specialization of the quantum parameters to specific values.*

For recent developments about this conjecture, see [**Co-Co-Ir-Ts**], [**Co-Ir-Ts**] and [**Co**]. Ruan's conjecture was formulated and largely studied on algebraic manifolds, as for instance on the weighted projective spaces [**Co-Co-Le-Ts**], [**Bo-Ma-Pe 1**], [**Bo-Ma-Pe 2**] (singularities are indeed difficult to deal with outside this context). A specially fruitful case is the case of the symmetric products of a smooth projective surface.

If X is a smooth complex projective surface and n a nonnegative integer, the punctual Hilbert scheme $X^{[n]}$ is the set of the 0-cycles of X of length n. A point ξ of $X^{[n]}$ is then an ideal sheaf $\mathcal{J}$ of $\mathcal{O}_X$ such that

– $\mathcal{O}_X/_{\mathcal{J}}$ is supported at a finite number of points $x_1,\dots,x_k$, which form the support of ξ.

– $\displaystyle\sum_{i=1}^{k}\ell_{x_i}(\xi)=n$, where $\ell_{x_i}(\xi)=\dim_{\mathbb{C}}\big(\mathcal{O}_X/_{\mathcal{J}}\big)_{|x_i}$ is the length of ξ at x_i.

The Hilbert scheme $X^{[n]}$ is a projective manifold.

The Hilbert-Chow morphism $HC\colon X^{[n]} \longrightarrow S^nX$ is defined by

$$HC(\xi) = \sum_{x \in \mathrm{supp}(\xi)} \ell_x(\xi)x.$$

This morphism is bijective over the dense stratum of S^nX formed of the n-tuples of distinct points. The two fundamental theorems on punctual Hilbert schemes of a projective surface are due to Fogarty and Briançon:

Theorem 2.17. [**Fo**] *If X is a smooth projective surface, then $X^{[n]}$ is smooth of dimension $2n$.*

Theorem 2.18. [**Br**] *If X is a smooth projective surface, $HC^{-1}(nx)$ is an irreducible algebraic variety of dimension $n-1$ for every x in X.*

In the case $n = 2$, $X^{[2]}$ can be described in an especially simple way: it is the quotient of the blowup of $X \times X$ along the diagonal by the natural action of $\mathbb{Z}/_{2\mathbb{Z}}$ consisting in switching the factors; the maximal fiber of the Hilbert-Chow morphism is then isomorphic to $\mathbb{P}^1$. If $n \geq 3$, this maximal fiber is irreducible by Briançon theorem, but it is no longer smooth.

The symmetric product S^nX is a singular Gorenstein projective manifold. Fogarty's theorem proves that $X^{[n]}$ is a resolution of the singularities of S^nX. This resolution is moreover crepant.

A long time before the crepant resolution conjecture was formulated, the Betti numbers and the punctual Hilbert schemes were computed by Göttsche (for a historical account of this formula, see [**Na 2**]).

Theorem 2.19. [**Gö**], [**Gö-So**] *If X is a quasi-projective smooth surface and if $n \in \mathbb{N}^*$, then*

$$\sum_{n=0}^{\infty} \sum_{i=0}^{4n} b_i\big(X^{[n]}\big) t^i q^n = \prod_{m=1}^{\infty} \frac{\big[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\big]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

This result can also be obtained as a consequence of a result due to Batyrev [**Ba**], which proves an additive version of the crepant conjecture.

After Göttsche's contribution, Nakajima's work [**Na 1**], then Lehn's [**Le**], Li, Qin and Wang's [**Li-Qi-Wa 2**], Lehn and Sorger's [**Le-So 1**], Fantechi and Göttsche's [**Fa-Gö**], have led to look deeply into the multiplicative structure of the cohomology rings $H^*\big(X^{[n]}, \mathbb{Q}\big)$. One of the achievements of these investigations has been the proof by Lehn and Sorger of the crepant conjecture for the punctual Hilbert schemes when X is a $K3$ surface.

Theorem 2.20. [**Le-So 2**] *If X is a $K3$ surface, there exists for every $n \in \mathbb{N}^*$ an isomorphism of graded rings*

$$H^*\big(X^{[n]}, \mathbb{Q}\big) \simeq H^*_{CR}\big(S^nX, \mathbb{Q}\big).$$

The study of Hilbert schemes went on beyond cohomology. The investigation of the class of $TX^{[n]}$ in K-theory by Ellinsgrud, Göttsche and Lehn has been a decisive step.

Theorem 2.21. [**El-Gö-Le**] *Let X be a projective smooth surface. The Chern numbers of $X^{[n]}$ can be computed by universal formulae starting from the Chern numbers of X.*

This theorem is equivalent to the following remarkable property:

The algebraic cobordism class of $X^{[n]}$ depends only on the algebraic cobordism class of X.

Thus we can see that the punctual Hilbert scheme provides an extraordinary example for the study of the crepant resolution conjecture. Nevertheless, the algebraic hypothesis is probably not crucial, and this conjecture seems rather of a symplectic kind.

Explicit examples of crepant resolutions in the almost-complex and symplectic settings are provided by the punctual Hilbert schemes $X^{[n]}$ of a symplectic or almost-complex fourfold which are desingularizations of the symmetric products S^nX. They were built by Voisin in [**Vo 1**] and [**Vo 2**].

Our aim in the second article is to study these almost-complex and symplectic punctual Hilbert schemes. To adapt to our context the classical theory on projective surfaces, we use instead of an integrable structure the relative integrable structures introduced in Voisin's construction.

DEFINITION 2.22. Let (X, J) be an almost complex fourfold. A *relative integrable structure* parametrized by S^nX is a family of integrable structures $\big(J_{\underline{x}}\big)_{\underline{x}\in S^nX}$ varying in a $\mathcal{C}^\infty$ way with $\underline{x}$ such that, for every $\underline{x}$ in S^nX, $J_{\underline{x}}$ is an integrable structure on an open subset $W_{\underline{x}}$ of X containing the points of $\underline{x}$.

Such structures exist and form, is they are chosen sufficiently close to J, a contractible set. Conversely, any relative integrable structure which is parametrized by a symmetric product S^nX defines an almost-complex structure on X. If J is such a structure, *Voisin's almost-complex Hilbert scheme* is defined as a set by

$$X^{[n]} = \bigsqcup_{\underline{x}\in S^nX} HC^{-1}_{W_{\underline{x}},\,J_{\underline{x}}}(\,\underline{x}\,)$$

where $HC_{W_{\underline{x}},\,J_{\underline{x}}} : W_{\underline{x}}^{[n]} \longrightarrow S^nW_{\underline{x}}$ is the Hilbert-Chow morphism associated to the integrable structures $J_{\underline{x}}$ on $W_{\underline{x}}$.

Here are now the main results we obtain in the paper "Topological properties of punctual Hilbert schemes of symplectic 4-dimensional manifolds". The first ones are only dealing with the additive structure of the cohomology ring of the almost-complex Hilbert schemes. In the case of smooth projective surfaces they are related to Göttsche's [**Gö**], [**Gö-So**], and Nakajima's, [**Na 1**] works.

THEOREM 2.23. *Let (X, J) be a an almost-complex compact fourfold.*

(i) *Let $\Big(b_i\big(X^{[n]}\big)\Big)_{i=0\ldots 4n}$ be the sequence of the Betti numbers of the almost-complex Hilbert scheme $X^{[n]}$. The generating series of the Betti numbers is given by*

$$\sum_{n=0}^{\infty}\sum_{i=0}^{4n} b_i\big(X^{[n]}\big)t^iq^n = \prod_{m=1}^{\infty}\frac{\big[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\big]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

(ii) *We can define Nakajima operators $q_i(\alpha)$, $i\in\mathbb{Z}$, $\alpha\in H^*(X,\mathbb{Q})$, in such a way that*
- *the following Nakajima relations are valid:*

$$\forall\, i,j\in\mathbb{Z},\quad \forall\,\alpha,\beta\in H^*(X,\mathbb{Q})\quad \big[q_i(\alpha),\, q_j(\beta)\big] = i\,\delta_{i+j,0}\int_X \alpha\beta,$$

- *if $\mathbb{H} = \bigoplus_{n=0}^{\infty} H^*\big(X^{[n]},\mathbb{Q}\big)$, the Nakajima operators define an irreducible highest weight representation of the Heisenberg algebra $\mathcal{H}\big(H^*(X,\mathbb{Q})\big)$ in $\mathbb{H}$ with highest weight vector 1.*

The Nakajima operators being built, we adapt Lehn's and Li-Qin-Wang's theory to obtain a description of the multiplicative structure of $H^*(X^{[n]},\mathbb{Q})$. The main step is the computation of the boundary operator ∂ defined by Lehn in [**Le**].

Definition 2.24. Let (X,J) be an almost-complex manifold and $X^{[n]}$ the associated almost-complex Hilbert scheme. The boundary operator $\partial:\mathbb{H}\longrightarrow\mathbb{H}$ is defined by

$$\forall\alpha\in H^*\big(X^{[n]},\mathbb{Q}\big),\quad \partial(\alpha)=-\frac{1}{2}\big[\partial X^{[n]}\big]\cup\alpha,$$

where $\partial X^{[n]}$ is the set of the schemes which are not supported at n distinct points and $\big[\partial X^{[n]}\big]$ is the associated cycle class in $H^2\big(X^{[n]},\mathbb{Q}\big)$.

We obtain the following result, which is due to Lehn in the integrable case:

Theorem 2.25. *Let (X,ω) be a symplectic compact fourfold and J an almost-complex structure adapted to ω. If ∂ is the boundary operator, then for all $n,m\in\mathbb{Z}$, and for all $\alpha,\beta\in H^*(X,\mathbb{Q})$, the following identity holds:*

$$\big[\partial\, q_n(\alpha)-q_n(\alpha)\partial, q_m(\beta)\big]=-nm\left\{q_{n+m}(\alpha\beta)-\frac{|n|-1}{2}\,\delta_{n+m,0}\left(\int_X c_1(X)\alpha\beta\right)\mathrm{id}\right\}.$$

We suppose that X is symplectic so that we can use to use Donaldson's results [**Do**]. Formal arguments due to Li-Qin-Wang [**Li-Qi-Wa 1**], [**Li-Qi-Wa 3**], [**Li-Qi-Wa 2**] and Lehn-Sorger [**Le-So 1**], [**Le-So 2**] allow us to obtain important consequences of the preceding theorem.

Theorem 2.26. *Let (X,ω) be a symplectic compact fourfold. We suppose that $b_1(X)=0$.*

(i) *The cohomology ring $H^*\big(X^{[n]},\mathbb{Q}\big)$ of the symplectic Hilbert scheme $X^{[n]}$ can be constructed by universal formulae starting from $H^*(X,\mathbb{Q})$, $c_1(X)$ and $c_2(X)$.*
(ii) *If the first Chern class of X is zero in $H^2(X,\mathbb{Q})$, then $H^*\big(X^{[n]},\mathbb{Q}\big)$ is isomorphic to the orbifold cohomology ring $H^*_{CR}\big(S^nX,\mathbb{Q}\big)$ of the symmetric product S^nX.*

The first part of the theorem is due in the algebraic setting to Li, Qin and Wang [**Li-Qi-Wa 3**].

Thus Ruan's crepant resolution conjecture happens to be true for the symmetric products of symplectic simply connected surfaces with first Chern class zero. This is a generalization of Lehn's and Sorger's result for $K3$ surfaces [**Le-So 2**].

All these statements deal only with the underlying topological space of $X^{[n]}$. Voisin's construction endows $X^{[n]}$ with structures which are richer than the structure of a topological space: $X^{[n]}$ is a differentiable manifold endowed with a stable almost-complex structure, and $X^{[n]}$ is symplectic if X is symplectic [**Vo 2**]. The tangent bundle $TX^{[n]}$ defines an element κ_n in the complex K-theory of $X^{[n]}$ and the almost-complex cobordism class of $X^{[n]}$ defined by the preceding complex structure is characterized by the Chern numbers of κ_n.

The Chern numbers of $X^{[n]}$ have been studied in the projective case by Ellinsgrud, Göttsche and Lehn in [**El-Gö-Le**], as seen above. They are obtained in a universal way starting from the Chern numbers of X. The proof of this result uses intensively the K-theory of coherent sheaves. We generalize this proof in the almost complex case, considering relative coherent sheaves on topological spaces fibered in analytic sets, which are associated to relative integrable structures. We obtain the

THEOREM 2.27. *Let* (X, J) *be an almost complex compact fourfold and* $X^{[n]}$ *the associated almost-complex Hilbert scheme. If* P *is a weighted polynomial in* $\mathbb{Q}\big[T_1, \ldots, T_{2n}\big]$ *of degree* $4n$*, where* T_i *is of degree* $2i$*, then there exists a weighted polynomial* $\widetilde{P} \in \mathbb{Q}\big[T_1, T_2\big]$ *of degree* 4*, depending only on* P*, such that*

$$\int_{X^{[n]}} P\big(c_1(X^{[n]}), \ldots, c_{2n}(X^{[n]})\big) = \int_X \widetilde{P}\big(c_1(X), c_2(X)\big).$$

This theorem means that the cobordism class of $X^{[n]}$ depends only on the cobordism class of X.

So Voisin's relative structures make it possible to adapt the classical theory to almost-complex or symplectic manifolds, which shows a posteriori that it is indeed in this context that punctual Hilbert schemes have to be considered. A sequel of this work could be the computation of the Gromov-Witten invariants for symplectic punctual Hilbert schemes; this problem is stated in [**Vo 2**].

Bibliographie

[Ab-Gr-Vi] D. Abramovich, T. Graber, A. Vistoli, Algebraic orbifold quantum products, *Orbifolds in Mathematics and Physics (Madison, WI, 2001)*, vol 310, Providence, RI: American Mathematical Society, (2002), 1–24.

[Ad-Le-Ru] A. Adem, J. Leida, Y. Ruan, Orbifolds and Stringy Topology, Cambridge University Press (2007).

[At] M. F. Atiyah, Complex analytic connections in fiber bundles, Trans. Amer. Math. Soc., *85*, (1957), 181–207.

[At-Hi] M. F. Atiyah, F. Hirzebruch, The Riemann-Roch theorem for Analytic embeddings, Topology, *1*, (1962), 151–166.

[Ba] V. V. Batyrev, Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs, J. Eur. Math. Soc., *1*, (1999), 5–33.

[Bei] A. A. Beilinson, Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra, Funkt. Analiz. Appl., *12(3)*, (1978), 68–69.

[Bo-Ma-Pe 1] S. Boissiere, E. Mann, F. Perroni, Crepant resolution of weighted projective spaces and quantum deformations, ArXiv:math/0610617.

[Bo-Ma-Pe 2] S. Boissiere, E. Mann, F. Perroni, The cohomological crepant resolution conjecture for $P(1,3,4,4)$, ArXiv:math/0712.3248.

[Bo-Se] A. Borel, J.-P. Serre, Le théorème de Riemann-Roch (d'après Grothendieck), Bull. Soc. Math. France, *86*, (1958), 97–136.

[Br] J. Briançon, Description de $\mathrm{Hilb}^n\mathbb{C}\{x,y\}$, Inventiones math., *41*, (1977), 45–89.

[Ca] H. Cartan, Idéaux et modules de fonctions analytiques de variables complexes, Bull. Soc. Math. France, *78*, (1950), 29–64.

[Ca-Se] H. Cartan, J.-P. Serre, Un théorème de finitude concernant les variétés analytiques compactes, Cr. Acad. Sci. Paris, *237*, (1953), 128–130.

[Ch-Ru 1] W. Chen, Y. Ruan, A new cohomology theory of orbifold, Comm. Math. Phys., *248:1*, (2004), 1–31.

[Ch-Ru 2] W. Chen, Y. Ruan, Orbifold Gromov-Witten theory, *Orbifolds in Mathematics and Physics (Madison, WI, 2001)*, vol 310, Providence, RI: American Mathematical Society, (2002), 25–85.

[Co] T. Coates, Wall-Crossings in Toric Gromov-Witten Theory II: Local examples, ArXiv:math/0804.2592.

[Co-Co-Ir-Ts] T. Coates, A. Corti, H. Iritani, H. Tseng, Computing genus-zero twisted Gromov-Witten invariants, ArXiv:math/0702234.

[Co-Co-Le-Ts] T. Coates, A. Corti, Y. Lee, H. Tseng, The Quantum Orbifold Cohomology of Weighted Projective Spaces, Preprint, ArXiv:math/0608481.

[Co-Ir-Ts] T. Coates, H. Iritani, H. Tseng, Wall-Crossings in Toric Gromov-Witten Theory I: Crepant examples, ArXiv:math/0611550.

[Co-Ru] T. Coates, Y. Ruan, Quantum cohomology and crepant resolutions: a conjecture, ArXiv:math/0710.5901.

[Do] S. Donaldson, Symplectic submanifolds and almost-complex geometry, J. Diff. Geom., *44*, (1996), 666–705.

[El-Gö-Le] G. Ellingsrud, L. Göttsche, M. Lehn, On the cobordism class of the Hilbert scheme of a surface, J. Algebraic Geom., *10:1*, (2001), 81–100.

[Es-Vi] H. Esnault, E. Viehweg, Deligne-Beilinson Cohomology, in *Beilinson's conjecture on special values of L-functions* (Rapoport, Schappacher & Schneider, eds.) Perspec. Math., Vol. 4, Academic Press, (1988), 43–91.

[Fa-Gö] B. Fantechi, L. Göttsche, Orbifold cohomology for global quotients, Duke Math. J., *117:2*, (2003), 197–227.

[Fo] J. Fogarty, Algebraic families on an algebraic surface, Am. J. Math., *10*, (1968), 511–521.

[Fu-Pa] W. Fulton, R. Pandharipande Notes on stable maps and quantum cohomology, Algebraic Geometry, Santa-Cruz, *62 II*, (1995), 45–96.

[Gö] L. Göttsche, The Betti numbers of the Hilbert of points on a smooth projective surface, Math. Ann., *286*, (1990), 193–207.

[Gö-So] L. Göttsche, W. Soergel, Perverse sheaves and the cohomology of Hilbert schemes on smooth algebraic surfaces, Math. Ann., *296*, (1993), 235–245.

[Gr-Re] H. Grauert, R. Remmert, Theory of Stein spaces, Springer (1979).

[Gr-Ri] H. Grauert, O. Riemenschneider, Verschwindungssätze für analytisch kohomologiegruppen auf Komplexen Räumen, Inv. Math., *136*, (1958), 236–292.

[Gre] H. I. Green, Chern classes for coherent sheaves, Ph.D. Thesis, University of Warwick, January 1980.

[Grom] M. Gromov, Pseudo holomorphic curves in symplectic manifolds, Invent. Math., *82*, (1985), 307–347.

[Grot] A. Grothendieck, La théorie des classes de Chern, Bull. Soc. Math. France, *86*, (1958), 137–154.

[Hiro] H. Hironaka, Flattening Theorem in Complex-Analytic Geometry, Amer. J. of Math., *97-2*, (1975), 503–547.

[Hirz] F. Hirzebruch, Topological methods in algebraic geometry, Springer, (1966).

[Ko-Va] J. Kock, I. Vainsencher An invitation to quantum cohomology, Birkhäuser (1999).

[Kod] K. Kodaira Complex manifolds and deformation of complex structures, Springer (1986).

[Ko-Ma] M. Kontsevich, Y. I. Manin Gromov-Witten classes, quantum cohomology and enumerative geometry, Comm. Math. Phys. *164*, (1994), 525–562.

[Le] M. Lehn, Chern classes of tautological sheaves on Hilbert schemes of points on sufaces, Invent. Math., *136*, (1999), 157–207.

[Le-So 1] M. Lehn, C. Sorger Symmetric groups and the cup product on the cohomology of Hilbert schemes, Duke Math. J., *110:2*, (2001), 345–357.

[Le-So 2] M. Lehn, C. Sorger The cup product of Hilbert schemes for $K3$ surfaces, Invent. Math., *152:2*, (2003), 305–329.

[Li-Qi-Wa 1] W.-P. Li, Z. Qin, W. Wang, Generators for the cohomology ring of Hilbert schemes of points on surfaces, Internat. Math. Res. Notices, *20*, (2001), 1057–1074.

[Li-Qi-Wa 2] W.-P. Li, Z. Qin, W. Wang, Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces, Math. Ann., *324:1*, (2002), 105–133.

[Li-Qi-Wa 3] W.-P. Li, Z. Qin, W. Wang, Stability of the cohomology rings of Hilbert schemes of points on surfaces, Reine Angew. Math., *554*, (2003), 217–234.

[McD-Sa] D. McDuff, D. Salamon, J-holomorphic Curves and Symplectic Topology, A.M.S. Colloquium Publications, Vol. 52, (2004).

[Na 1] H. Nakajima, Heisenberg algebra and Hilbert schemes on projective surfaces, Ann. Math, *145*, (1997), 379–388.

[Na 2] H. Nakajima, Lectures on Hilbert schemes of points on surfaces, American Mathematical Society, Volume 18, (1999).

[OB-To-To] N. R. O'Brian, D. Toledo, Y. L. L. Tong, A Grothendieck-Riemann-Roch formula for maps of complex manifolds, Math. Ann., *271*, (1985), 493–526.

[Ro] H. Rossi, Picard variety of an isolated singular point, Rice Univ. Studies, , *54-4*, (1968), 63–73.

[Qi-Wa] Z. Qin, W Wang, Hilbert schemes and symmetric products: a dictionary, *Orbifolds in Mathematics and Physics (Madison, WI, 2001)*, Contemp. Math., *310*, Americam Mathematical Society (2002), 233–257.

[Ru-Ti] Y. Ruan, G Tian, A mathematical theory of quantum cohomology, J. Differential Geometry, *42-2*, (1995), 259–367.

[Se 1] J.-P. Serre, Faiceaux algébriques cohérents, Ann. of Math, *61*, (1955), 197–278

[Se 2] J.-P. Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier (Grenoble), *6*, (1956) 1–42.

[To-To 1] D. Toledo, Y. L. L. Tong, A parametrix for $\overline{\partial}$ and Riemann-Roch in Čech theory, Topology, *15*, (1976), 273–301.

[To-To 2] D. Toledo, Y. L. L. Tong, Green's theory of Chern classes and the Riemann-Roch formula, Contemp. Math., *58-I*, (1986), 261–275.

[Vo 1] C. Voisin, On the Hilbert scheme of points on an almost complex fourfold, Ann. Inst. Fourier, *50,2*, (2000), 689–722.

[Vo 2] C. Voisin, On the punctual Hilbert scheme of a symplectic fourfold, Contemporary Mathematics, (2002), 265–289.

[Vo 3] C. Voisin, Théorie de Hodge et géométrie algébrique complexe, Société Mathématique de France, (2002).

[Vo 4] C. Voisin, A counterexample to the Hodge conjecture extended to Kähler varieties, Int. Math. Res. Notices, *20*, (2002), 1057–1075.

CHAPITRE 2

Classes de Chern en cohomologie de Deligne pour les faisceaux analytiques cohérents

Dans ce chapitre, nous présentons une version détaillée de l'article « Chern classes in Deligne cohomology for coherent analytic sheaves ». Des compléments sur la cohomologie de Deligne sont fournis dans le chapitre 3.

Chern classes in Deligne cohomology for coherent analytic sheaves

ABSTRACT. — In this article, we construct Chern classes in rational Deligne cohomology for coherent sheaves on a smooth complex compact manifold. We prove that these classes satisfy the functoriality property under pullbacks, the Whitney formula and the Grothendieck-Riemann-Roch theorem for projective morphisms between smooth complex compact manifolds.

1. Introduction

Let X be a smooth differentiable manifold and E be a complex vector bundle of rank r on X. The Chern-Weil theory (see [**Gri-Ha**, Ch. 3 § 3]) constructs classes $c_i(E)^{\mathrm{top}}$, $1 \leq i \leq r$, with values in the de Rham cohomology $H^{2i}(X, \mathbb{R})$, which generalize the first Chern class of a line bundle in $H^2(X, \mathbb{Z})$ obtained by the exponential exact sequence. These classes are compatible with pullbacks under smooth morphisms and verify the Whitney sum formula

$$c_k(E \oplus F)^{\mathrm{top}} = \sum_{i+j=k} c_i(E)^{\mathrm{top}} \, c_j(F)^{\mathrm{top}}.$$

There exist more refined ways of defining $c_i(E)^{\mathrm{top}}$ in $H^{2i}(X, \mathbb{Z})$. The first method is due to Chow (see the introduction of [**Gro 1**]). The idea is to define explicitly the Chern classes of the universal bundles of the grassmannians and to write any complex vector bundle as a quotient of a trivial vector bundle. Of course, computations have to be done on the grassmannians to check the compatibilities. Note that in the holomorphic or in the algebraic context, a vector bundle is not in general a quotient of a trivial vector bundle. Nevertheless, if X is projective, this is true after tensorising by a sufficiently high power of an ample line bundle and the construction can be adapted (see [**Br**]).

A more intrinsic construction is the splitting method, introduced by Grothendieck in [**Gro 1**]. Let us briefly recall how it works. By the Leray-Hirsh theorem, we know that $H^*(\mathbb{P}(E),\mathbb{Z})$ is a free module over $H^*(X,\mathbb{Z})$ with basis 1, $\alpha,\dots,\alpha^{r-1}$, where α is the opposite of the first Chern class of the relative Hopf bundle on $\mathbb{P}(E)$. Now the Chern classes of E are uniquely defined by the relation

$$\alpha^r + p^*c_1(E)^{\mathrm{top}}\,\alpha^{r-1} + \cdots + p^*c_{r-1}^{\mathrm{top}}(E)\,\alpha + p^*c_r(E)^{\mathrm{top}} = 0.$$

(see Grothendieck [**Gro 1**], Voisin [**Vo 1**, Ch. 11 § 2], and Zucker [**Zu**, § 1]).

The splitting method works amazingly well in various contexts, provided that we have

- the definition of the first Chern class of a line bundle,
- a structure theorem for the cohomology ring of a projective bundle considered as a module over the cohomology ring of the base.

Let us now examine the algebraic case. Let X be a smooth algebraic variety over a field k of characteristic zero, and E be an algebraic bundle on X. Then the splitting principle allows to define $c_i(E)^{\mathrm{alg}}$

- in the Chow ring $CH^i(X)$ if X is quasi-projective,
- in the algebraic de Rham cohomology group $H^{2i}_{\mathrm{DR}}(X/k)$.

Suppose now that $k=\mathbb{C}$. Then Grothendieck's comparison theorem (see [**Gro 3**]) says that we have a canonical isomorphism between $H^{2i}_{DR}(X/\mathbb{C})$ and $H^{2i}(X^{\mathrm{an}},\mathbb{C})$. It is important to notice that the class $c_i(E)^{\mathrm{alg}}$ is mapped to $(2\pi\sqrt{-1})^i c_i(E)^{\mathrm{top}}$ by this morphism.

Next, we consider the problem in the abstract analytic setting. Let X be a smooth complex analytic manifold and E be a holomorphic vector bundle on X. We denote by $\mathcal{A}^{p,q}_{\mathbb{C}}(X)$ the space of complex differential forms of type (p,q) on X and we put $\mathcal{A}_{\mathbb{C}}(X)=\bigoplus_{p,q}\mathcal{A}^{p,q}_{\mathbb{C}}(X)$. The Hodge filtration on $\mathcal{A}_{\mathbb{C}}(X)$ is defined by $F^i\mathcal{A}_{\mathbb{C}}(X)=\bigoplus_{p\geq i,q}\mathcal{A}^{p,q}_{\mathbb{C}}(X)$. It induces a filtration $F^iH^k(X,\mathbb{C})$ on $H^k(X,\mathbb{C})$. For a detailed exposition see [**Vo 1**, Ch. 7 and 8]. Let $\Omega^\bullet_X$ be the holomorphic de Rham complex on X. This is a complex of locally free sheaves. We can consider the analytic de Rham cohomology $\mathbb{H}^{k+i}(X,\Omega_X^{\bullet\geqslant i})$ which is the hypercohomology of the truncated de Rham complex. The maps of complexes $\Omega_X^{\bullet\geqslant i}\longrightarrow\Omega^\bullet_X$ and $\Omega_X^{\bullet\geqslant i}\longrightarrow\Omega^i_X[-i]$ give two maps $\mathbb{H}^{k+i}(X,\Omega_X^{\bullet\geqslant i})\longrightarrow F^iH^k(X,\mathbb{C})$ and $\mathbb{H}^{k+i}(X,\Omega_X^{\bullet\geqslant i})\longrightarrow H^k(X,\Omega^i_X)$. In the compact Kähler case, the first map is an isomorphism, but it is no longer true in the general case. We will denote by $H^{p,q}(X)$ the cohomology classes in $H^{p+q}(X,\mathbb{C})$ which admit a representative in $\mathcal{A}^{p,q}(X)$.

If E is endowed with the Chern connection associated to a hermitian metric, the de Rham representative of $c_i(E)^{\mathrm{top}}$ obtained by Chern-Weil theory is of type (i,i) and is unique modulo $d(F^i\mathcal{A}_X^{2i-1})$. This allows to define $c_i(E)$ in $\mathbb{H}^{2i}(X,\Omega_X^{\bullet\geqslant i})$, and then in $H^{i,i}(X)$ and $H^i(X,\Omega^i_X)$. The notations for these three classes will be $c_i(E)^{\mathrm{an}}$, $c_i(E)^{\mathrm{hodge}}$ and $c_i(E)^{\mathrm{dolb}}$.

If we forget the holomorphic structure of E, we can consider its topological Chern classes $c_i(E)^{\mathrm{top}}$ in the Betti cohomology groups $H^{2i}(X,\mathbb{Z})$. The image of $c_i(E)^{\mathrm{top}}$ in $H^{2i}(X,\mathbb{C})$ is $c_i(E)^{\mathrm{hodge}}$. Thus $c_i(E)^{\mathrm{top}}$ is an integral cohomology class whose image in $H^{2i}(X,\mathbb{C})$ lies in $F^iH^{2i}(X,\mathbb{C})$. Such classes are called *Hodge classes* of weight $2i$.

The Chern classes of E in $H^{2i}(X,\mathbb{Z})$, $F^iH^{2i}(X,\mathbb{C})$, $\mathbb{H}^{2i}\big(X,\Omega_X^{\bullet\geqslant i}\big)$ and $H^i\big(X,\Omega_X^i\big)$ appear in the following diagram:

$$\begin{tikzcd}
 & H^{2i}(X,\mathbb{Z}) \ni c_i(E)^{\mathrm{top}} \arrow[d] \\
c_i(E)^{\mathrm{an}} \in \mathbb{H}^{2i}(X,\Omega_X^{\bullet\geqslant i}) \arrow[r] \arrow[d] & F^iH^{2i}(X,\mathbb{C}) \ni c_i(E)^{\mathrm{hodge}} \\
c_i(E)^{\mathrm{dolb}} \in H^i(X,\Omega_X^i) &
\end{tikzcd}$$

This means in particular that these different classes are compatible with the Hodge decomposition in the compact Kähler case, and in general via the Hodge $\longrightarrow$ de Rham spectral sequence. Furthermore, the knowledge of $c_i(E)^{\mathrm{an}}$ allows to obtain the two other classes $c_i(E)^{\mathrm{dolb}}$ and $c_i(E)^{\mathrm{hodge}}$, but the converse is not true. Thus $c_i(E)^{\mathrm{an}}$ contains more information (except torsion) than the other classes in the diagram.

Recall now the Deligne cohomology groups $H^p_{\mathrm{Del}}(X,\mathbb{Z}(q))$ (see [**Es-Vi**, § 1] and Section 3.1). We will be mainly interested in the cohomology groups $H^{2i}_{\mathrm{Del}}(X,\mathbb{Z}(i))$. They admit natural maps to $H^{2i}(X,\mathbb{Z})$ and to $\mathbb{H}^{2i}(X,\Omega_X^{\bullet\geqslant i})$, which are compatible with the diagram above. Furthermore, there is an exact sequence

$$0 \longrightarrow \mathbb{H}^{2i-1}(X,\Omega_X^{\bullet\leqslant i-1})\big/_{H^{2i-1}(X,\mathbb{Z})} \longrightarrow H^{2i}_{\mathrm{Del}}(X,\mathbb{Z}(i)) \longrightarrow H^{2i}(X,\mathbb{Z}),$$

(see [**Vo 1**] and Proposition 3.3 (i)). Thus a Deligne class is a strong refinement of any of the above mentioned classes. The splitting method works for the construction of $c_i(E)$ in $H^{2i}_{\mathrm{Del}}(X,\mathbb{Z}(i))$, as explained in [**Zu**, § 4], and [**Es-Vi**, § 8]. These Chern classes, as all the others constructed above, satisfy the following properties:

- they are functorial with respect to pullbacks.
- if $0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$ is an exact sequence of vector bundles, then for all i
$$c_i(F) = \sum_{p+q=i} c_p(E)\, c_q(G).$$

The last property means that the total Chern class $c = 1 + c_1 + \cdots + c_n$ is defined on the Grothendieck group $K(X)$ of holomorphic vector bundles on X and satisfies the additivity property $c(x+x') = c(x)c(x')$.

Now, what happens if we work with coherent sheaves instead of locally free ones? If X is quasi-projective and $\mathcal{F}$ is an algebraic coherent sheaf on X, there exists a locally free resolution

$$0 \longrightarrow E_1 \longrightarrow \cdots \longrightarrow E_N \longrightarrow \mathcal{F} \longrightarrow 0.$$

(This is still true under the weaker assumption that X is a regular separated scheme over $\mathbb{C}$ by Kleiman's lemma; see [**SGA 6**, II, 2.2.7.1]). The total Chern class of $\mathcal{F}$ is defined by

$$c(\mathcal{F}) = c(E_N)\, c(E_{N-1})^{-1} c(E_{N-2}) \dots$$

The class $c(\mathcal{F})$ does not depend on the locally free resolution (see [**Bo-Se**, § 4 and 6]). More formally, if $K(X)$ is the Grothendieck group of coherent sheaves on X, the canonical map $\iota\colon K(X) \longrightarrow K(X)$ is an isomorphism. The inverse is given by

$$[\mathcal{F}] \longrightarrow [E_N] - [E_{N-1}] + [E_{N-2}] - \dots$$

In what follows, we consider the complex analytic case. The problem of the existence of global locally free resolutions in the analytic case has been open for a long time. For smooth complex surfaces, such resolutions always exist by [**Sch**]. More recently, Schröer and Vezzosi proved in [**ScVe**] the same result for singular separated surfaces. Nevertheless, for varieties of dimension at least 3, a negative answer to the question is provided by the following counterexample of Voisin:

THEOREM 1.1. [**Vo 2**] *On any generic complex torus of dimension at least* 3*, the ideal sheaf of a point does not admit a global locally free resolution.*

Worse than that, even if $\mathcal{F}$ admits a globally free resolution $E^{\bullet}$, the method of Borel and Serre [**Bo-Se**] does not prove that $c(E_N)\, c(E_{N-1})^{-1} c(E_{N-2}) \dots$ is independent of $E^{\bullet}$. In fact, the crucial point in their argument is that *every* coherent sheaf should have a resolution.

Nevertheless Borel-Serre's method applies in a weaker context if we consider Chern classes in $H^*(X, \mathbb{Z})$. Indeed, if $\mathcal{F}$ is a coherent sheaf on X and $\mathcal{C}^{\omega}_X$ is the sheaf of real-analytic functions on X, then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{C}^{\omega}_X$ admits a locally free real-analytic resolution by the Grauert vanishing theorem [**Gra**]. We obtain by this method topological Chern classes $c_i(\mathcal{F})^{\mathrm{top}}$ in $H^{2i}(X, \mathbb{Z})$.

It is natural to require that c_i should take its values in more refined rings depending on the holomorphic structure of $\mathcal{F}$ and X. Such a construction has been carried out by Atiyah for the Dolbeault cohomology ring in [**At**]. Let us briefly describe his method: the exact sequence

$$0 \longrightarrow \mathcal{F} \otimes \Omega^1_X \longrightarrow \wp^1_X(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow 0$$

of principal parts of $\mathcal{F}$ of order one (see [**EGA IV**, § 16.7]) gives an extension class (the Atiyah class) $a(\mathcal{F})$ in $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F} \otimes \Omega^1_X)$. Then $c_p(\mathcal{F})$ is the trace of the p-th Yoneda product of $a(\mathcal{F})$. These classes are used by O'Brian, Toledo and Tong in [**OB-To-To 1**] and [**OB-To-To 2**] to prove the Grothendieck-Riemann-Roch theorem on abstract manifolds in the Hodge ring. The Atiyah class has been constructed by Grothendieck and Illusie for perfect complexes (see [**Ill**, Ch. 5]). Nevertheless, if X is not a Kähler manifold, there is no good relation between $H^p(X, \Omega^p_X)$ and $\mathbb{H}^{2p}(X, \Omega^{\bullet \geqslant p}_X)$, as the Frölicher spectral sequence may not degenerate at E_1 for example.

In this context, the most satisfactory construction was obtained by Green in his unpublished thesis (see [**Gre**] and [**To-To**]). He proved the following theorem

THEOREM 1.2. [**Gre**], [**To-To**] *Let $\mathcal{F}$ be a coherent sheaf on X. Then there exist Chern classes $c_i(\mathcal{F})^{\mathrm{Gr}}$ in the analytic de Rham cohomology groups $\mathbb{H}^{2i}(X, \Omega^{\bullet \geqslant i}_X)$ which are compatible with Atiyah Chern classes and topological Chern classes.*

In order to avoid the problem of nonexistence of locally free resolutions, he introduced the notion of a *simplicial resolution* by simplicial vector bundles with respect to a given covering. Green's basic result is the following:

THEOREM 1.3. [**Gre**], [**To-To**] *Any coherent sheaf on a smooth complex compact manifold admits a finite simplicial resolution by simplicial holomorphic vector bundles.*

The next step in order to obtain Theorem 1.2 above, is to define the Chern classes of a simplicial vector bundle. For this, Green uses Bott's construction (see [**Bott**]) which can be adapted to the simplicial context. Though, it is not clear how to extend Green's method to Deligne cohomology.

Let us now state the main result of this article:

THEOREM 1.4. *Let X be a complex compact manifold. For every coherent sheaf $\mathcal{F}$ on X, we can define a Chern character* $\mathrm{ch}(\mathcal{F})$ *in* $\oplus_p H^{2p}_{\mathrm{Del}}(X, \mathbb{Q}(p))$ *such that:*

(i) *For every exact sequence* $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ *of coherent sheaves on X, we have* $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$. *The induced Chern character*

$$\mathrm{ch}\colon K(X) \longrightarrow H^{*}_{\mathrm{Del}}(X, \mathbb{Q})$$

is a ring morphism.

(ii) *If* $f\colon X \longrightarrow Y$ *is holomorphic and if* $f^{\dagger}\colon K(Y) \longrightarrow K(X)$ *is the pullback morphism in analytic K-theory (see Section 2), then for any element y in $K(Y)$,* $\mathrm{ch}(f^{\dagger} y) = f^{*}\,\mathrm{ch}(y)$.

(iii) *If $\mathcal{E}$ is a locally free sheaf, then* $\mathrm{ch}(\mathcal{E})$ *is the usual Chern character in rational Deligne cohomology.*

(iv) *If Z is a smooth closed submanifold of X and $\mathcal{F}$ a coherent sheaf on Z, the Grothendieck-Riemann-Roch (GRR) theorem is valid for* $\big(i_Z, \mathcal{F}\big)$, *namely*

$$\mathrm{ch}\big(i_{Z*}\mathcal{F}\big) = i_{Z*}\Big(\mathrm{ch}(\mathcal{F})\ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big).$$

(v) *If* $f\colon X \longrightarrow Y$ *is a projective morphism between smooth complex compact manifolds, for every coherent sheaf $\mathcal{F}$ on X, we have the Grothendieck-Riemann-Roch theorem*

$$\mathrm{ch}\big(f_{!}[\mathcal{F}]\big)\,\mathrm{td}(Y) = f_{*}[\mathrm{ch}(\mathcal{F})\,\mathrm{td}(X)].$$

Here, $f_{!}$ is the Gysin morphism in analytic K-theory (see Section 2) and f_{*} is the Gysin morphism in Deligne cohomology (see Section 3.1 and Section 7 of Chapter 3)

Our approach is completely different from [**Bo-Se**]. Indeed, Voisin's result prevents from using locally free resolutions. Our geometric starting point, which will be used instead of locally free resolutions, is the following (Theorem 4.10):

THEOREM 1.5. [**Ro**] *Let $\mathcal{F}$ be a coherent sheaf on X of generic rank r. Then there exists a bimeromorphic morphism* $\pi\colon \widetilde{X} \longrightarrow X$ *and a locally free sheaf $\mathcal{Q}$ on $\widetilde{X}$ of rank r, together with a surjective map* $\pi^{*}\mathcal{F} \longrightarrow \mathcal{Q}$.

It follows that up to torsion sheaves, $\pi^{\dagger}[\mathcal{F}]$ is locally free. This will allow us to define our Chern classes by induction on the dimension of the base. Of course, we will need to show that our Chern classes satisfy the Whitney formula and are independent of the bimeromorphic model $\widetilde{X}$.

The theorem above is a particular case of Hironaka's flattening theorem (see [**Hiro 2**] and in the algebraic case [**Gr-Ra**]). Indeed, if we apply Hironaka's result to the couple $(\mathcal{F}, \mathrm{id})$, there exists a bimeromorphic map $\sigma\colon \widetilde{X} \longrightarrow X$ such that $\sigma^{*}\mathcal{F} \,/\, \big(\sigma^{*}\mathcal{F}\big)_{\mathrm{tor}}$ is flat with respect to the identity morphism, and thus locally free. For the sake of completeness, we include an elementary proof of Theorem 4.10, which is similar to the proof of [**Ro**].

Property (iv) of Theorem 1.4 is noteworthy. The lack of global resolutions (see [**Vo 2**]) prevents from using the proofs of Borel, Serre and of Baum, Fulton and McPherson (see [**Fu**, Ch. 15 § 2]). The equivalent formula in the topological setting is proved in [**At-Hi**]. In the holomorphic context, O'Brian, Toledo and Tong ([**OB-To-To 3**]) prove this formula for the Atiyah Chern classes when there exists a retraction from X to Z, then they establish (GRR) for a projection and they deduce that (GRR) is valid for any holomorphic map, so *a posteriori* for an immersion (see [**OB-To-To 2**]). Nevertheless, our result does not give a new proof of (GRR) formula for an immersion in the case of the Atiyah Chern classes. Indeed, the compatibility between our construction and the Atiyah Chern classes is a consequence of the (GRR) theorem for an immersion in both theories, as explained further.

Property (v) is an immediate consequence of (iv), as originally noticed in [**Bo-Se**], since the natural map from $K(X) \otimes_{\mathbb{Z}} K(\mathbb{P}^N)$ to $K(X \times \mathbb{P}^N)$ is surjective (see [**SGA 6**, Exposé VI] and [**Bei**]). Yet, we do not obtain the (GRR) theorem for a general holomorphic map between smooth complex compact manifolds.

Remark that $c_p(\mathcal{F})$ is constructed in the rational Deligne cohomology group $H^{2p}_{\mathrm{Del}}(X, \mathbb{Q}(p))$ and not in $H^{2p}_{\mathrm{Del}}(X, \mathbb{Z}(p))$. The reason is that we make full use of the Chern character, which has denominators, and thus determines the total Chern class only up to torsion classes. We think that it could be possible to define $c_p(\mathcal{F})$ in $H^{2p}_{\mathrm{Del}}(X, \mathbb{Z}(p))$ following our approach, but with huge computations. For $p = 1$, $c_1(\mathcal{F})$ can be easily constructed in $H^2_{\mathrm{Del}}(X, \mathbb{Z}(1))$. Indeed, it suffices to define $c_1(\mathcal{F}) = c_1(\det \mathcal{F})$, where $\det \mathcal{F}$ is the determinant line bundle of $\mathcal{F}$ (see [**Kn-Mu**]).

It is interesting to compare the classes of Theorem 1.4 with other existing theories. This is the purpose of Theorem 4.10. We prove that, for any cohomology ring satisfying reasonable properties, a theory of Chern classes can be completely determined if we suppose that the GRR formula is valid for immersions. More precisely, our statement is the following (Theorem 6.5):

Theorem 1.6. *Under the hypotheses* (α)-(δ) *of page* 84 *on the cohomology ring, a theory of Chern classes for coherent sheaves on smooth complex compact manifolds which satisfies the GRR theorem for immersions, the Whitney additivity formula and the functoriality formula is completely determined by the first Chern class of holomorphic line bundles.*

This theorem yields compatibility results (Corollary 6.7):

Corollary 1.7. *The classes of Theorem* 1.4 *are compatible with the rational topological Chern classes and the Atiyah Chern classes.*

Nevertheless, since GRR for immersions does not seem to be known for the Green Chern classes if X is not Kähler, the theorem above does not give the compatibility in this setting. In fact, the compatibility is *equivalent* to the GRR theorem for immersions for the Green Chern classes.

Let us mention the link of our construction with secondary characteristic classes.

We can look at a subring of the ring of Cheeger-Simons characters on X which are the "holomorphic" characters (that is the G-cohomology defined in [**Es**, § 4], or equivalently the restricted differential characters defined in [**Br**, § 2]). This subring can be mapped onto the Deligne cohomology ring, but not in an injective way in general. When E is a holomorphic vector bundle with a compatible connection, the Cheeger-Simons theory (see [**Ch-Si**]) produces Chern classes with values in this subring. It is known that these classes are the same as the Deligne classes (see [**Br**] in the algebraic case and [**Zu**, § 5] for the general case). When E is topologically trivial, this construction gives the so-called secondary classes with values in the intermediate jacobians of X (see [**Na**] for a different construction, and [**Ber**] who proves the link with the generalized Abel-Jacobi map). The intermediate jacobians of X have been constructed in the Kähler case by Griffiths. They are complex tori (see [**Vo 1**, Ch. 12 § 1], and [**Es-Vi**, § 7 and 8]). If X is not Kähler, intermediate jacobians can still be defined but they are no longer complex tori.

Our result provides similarly refined Chern classes for coherent sheaves, and in particular, secondary invariants for coherent sheaves with trivial topological Chern classes.

The organization of the paper is the following. We recall in Part 3 the basic properties of Deligne cohomology and Chern classes for locally free sheaves in Deligne cohomology. The necessary results of analytic K-theory with support are grouped in Appendix 7; they will be used extensively throughout the paper. The rest of the article is devoted to the proof of Theorem 1.4. The construction of the Chern classes is achieved by induction on $\dim X$. In Part 4 we perform the induction step for torsion sheaves using the (GRR) formula for the immersion of smooth divisors; then we prove a dévissage theorem which enables us to break any coherent sheaf into a locally free sheaf and a torsion sheaf on a suitable modification of X; this is the key of the construction of $c_i(\mathcal{F})$ when $\mathcal{F}$ has strictly positive rank. The Whitney formula, which is a part of the induction process, is proved in Part 5. After several reductions, we use a deformation argument which leads to the deformation space of the normal cone of a smooth hypersurface. We establish the (GRR) theorem for the immersion of a smooth hypersurface in Part 4, in Part 6 we recall how to deduce the general (GRR) theorem for an immersion from this particular case, using excess formulae, then we deduce uniqueness results using Theorem 4.10.

Acknowledgement. I wish to thank Claire Voisin for introducing me to this beautiful subject and for many helpful discussions. I also thank Pierre Schapira for enlightening conversations.

2. Notations and conventions

All manifolds are complex smooth analytic connected manifolds. All the results are clearly valid for non connected ones, by reasoning on the connected components.

Except in Section 3, all manifolds are compact. By *submanifold*, we always mean a *closed* submanifold.

Holomorphic vector bundles

The rank of a holomorphic vector bundle is well defined since the manifolds are connected. If E is a holomorphic vector bundle, we denote by $\mathcal{E}$ the associated locally free sheaf. The letter $\mathcal{E}$ always denotes a locally free sheaf.

Coherent sheaves

If $\mathcal{F}$ is a coherent analytic sheaf on a smooth connected manifold X, then $\mathcal{F}$ is locally free outside a proper analytic subset S of X (see [**Gra-Re**]). Then $U = X\backslash S$ is connected. By definition, the generic rank of $\mathcal{F}$ is the rank of the locally free sheaf $\mathcal{F}_{|U}$. If the generic rank of $\mathcal{F}$ vanishes, $\mathcal{F}$ is supported in a proper analytic subset Z of X, it is therefore annihilated by the action of a sufficiently high power of the ideal sheaf $\mathcal{I}_Z$. Conversely, if $\mathcal{F}$ is a torsion sheaf, $\mathcal{F}$ is identically zero outside a proper analytic subset of X, so it has generic rank zero. The letter $\mathcal{T}$ always denotes a torsion sheaf.

Divisors

A strict normal crossing divisor D in X is a formal sum $m_1 D_1 + \cdots + m_N D_N$, where D_i, $1 \leq i \leq N$, are smooth transverse hypersurfaces and m_i, $1 \leq i \leq N$, are nonzero integers. If all the coefficients m_i are positive, D is *effective*. In that case, the associated reduced divisor D^{red} is the effective divisor $D_1 + \cdots + D_N$. A strict simple normal crossing divisor is *reduced* if for all i, $m_i = 1$. We make no difference between a reduced divisor and its support.

By a *normal crossing divisor* we always mean a *strict* normal crossing divisor. If D is an effective simple normal crossing divisor, it defines an ideal sheaf $\mathcal{I}_D = \mathcal{O}_X(-D)$. The associated quotient sheaf is denoted by $\mathcal{O}_D$.

We use frequently Hironaka's desingularization theorem [**Hiro 1**] for complex spaces as stated in [**An-Ga**, Th. 7.9 and 7.10].

Tor sheaves

Let $f : X \longrightarrow Y$ be a holomorphic map and $\mathcal{F}$ be a coherent sheaf on Y. We denote by $\mathrm{Tor}_i(\mathcal{F}, f)$ the sheaf $\mathrm{Tor}_i^{f^{-1}\mathcal{O}_Y}\big(f^{-1}\mathcal{F}, \mathcal{O}_X\big)$.

Grothendieck groups

The Grothendieck group of coherent analytic sheaves (resp. of torsion coherent analytic sheaves) on a complex space X is denoted by $K(X)$ (resp. $K_{\mathrm{tors}}(X)$). If $\mathcal{F}$ is a coherent analytic sheaf on X, $[\mathcal{F}]$ denotes its class in $K(X)$. The notation $K_Z(X)$ is defined in Appendix 7. In order to avoid subtle confusions, we never use here the Grothendieck group of locally free sheaves.

If $f : X \longrightarrow Y$ is a holomorphic map and $\mathcal{F}$ is a coherent sheaf on Y, the element $f^{\dagger}[\mathcal{F}]$ in $K(X)$ is defined by $f^{\dagger}[\mathcal{F}] = \sum_{i\geq 0}(-1)^i\big[\mathrm{Tor}_i(\mathcal{F}, f)\big]$. The pullback map $f^{\dagger} : K(Y) \longrightarrow K(X)$ is a ring morphism. For the analogous pullback map for analytic K-theory with support, see Appendix 7.3.

If $f : X \longrightarrow Y$ is a proper holomorphic map and $\mathcal{G}$ is a coherent sheaf on X, the element $f_![\mathcal{G}]$ in $K(Y)$ is defined by $f_![\mathcal{G}] = \sum_{i\geq 0}(-1)^i[R^i f_* \mathcal{G}]$. The Gysin morphism $f_! : K(X) \longrightarrow K(Y)$ satisfies the projection formula $f_!(x \,.\, f^{\dagger} y) = f_! x \,.\, y$.

3. Deligne cohomology and Chern classes for locally free sheaves

In this section, we will expose the basics of Deligne cohomology for the reader's convenience. For a more detailed exposition, see [**Es-Vi**, § 1, 6, 7, 8], [**Vo 1**, Ch. 12], and [**EZZ**].

3.1. Deligne cohomology.

Definition 3.1. Let X be a smooth complex manifold and let p be a nonnegative integer. Then

- The Deligne complex $\mathbb{Z}_{D,X}(p)$ of X is the following complex of sheaves
$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{(2i\pi)^p} \mathcal{O}_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^{p-1},$$
where $\mathbb{Z}_X$ is in degree zero. Similarly, the rational Deligne complex $\mathbb{Q}_{D,X}(p)$ is the same complex as above with $\mathbb{Z}_X$ replaced by $\mathbb{Q}_X$.
- The Deligne cohomology groups $H^i_{\mathrm{Del}}(X, \mathbb{Z}(p))$ are the hypercohomology groups defined by
$$H^i_{\mathrm{Del}}(X, \mathbb{Z}(p)) = \mathbb{H}^i(X, \mathbb{Z}_{D,X}(p)).$$
The rational Deligne cohomology groups are defined by the same formula as the hypercohomology groups of the rational Deligne complex.
- The same definition holds for the Deligne cohomology with support in a closed subset Z:
$$H^i_{\mathrm{Del},Z}(X, \mathbb{Z}(p)) = \mathbb{H}^i_Z(X, \mathbb{Z}_{D,X}(p)).$$
- The total Deligne cohomology group of X is $H^*_{\mathrm{Del}}(X) = \bigoplus\limits_{k,p} H^k_{\mathrm{Del}}(X, \mathbb{Z}(p))$. We will denote by $H^*_{\mathrm{Del}}(X, \mathbb{Q})$ the total rational Deligne cohomology group.

Example 3.2.

- $H^i_{\mathrm{Del}}(X, \mathbb{Z}(0))$ is the usual Betti cohomology group $H^i(X, \mathbb{Z})$.
- $\mathbb{Z}_{D,X}(1)$ is quasi-isomorphic to $\mathcal{O}^*_X[-1]$ by the exponential exact sequence. Thus we have a group isomorphism $H^2_{\mathrm{Del}}(X, \mathbb{Z}(1)) \simeq H^1\big(X, \mathcal{O}^*_X\big) \simeq \mathrm{Pic}(X)$. The first Chern class of a line bundle L in $H^2_{\mathrm{Del}}(X, \mathbb{Z}(1))$ is the element of $\mathrm{Pic}(X)$ defined by $c_1(L) = \{L\}$.

– $H^2_{\mathrm{Del}}(X,\mathbb{Z}(2))$ is the group of flat holomorphic line bundles, i.e. holomorphic line bundles with a holomorphic connection (see [**Es-Vi**, § 1] and [**Es**]).

For geometric interpretations of higher Deligne cohomology groups, we refer the reader to [**Ga**].

Some fundamental properties of Deligne cohomology are listed below (for the proof as well as detailed computations for the cup-product, see Chapter 3, Sections 2 and 3):

Proposition 3.3.

(i) *We have an exact sequence* $0 \longrightarrow \Omega_X^{\bullet\leqslant p-1}[-1] \longrightarrow \mathbb{Z}_{D,X}(p) \longrightarrow \mathbb{Z}_X \longrightarrow 0$.

In particular, $H^{2p}_{\mathrm{Del}}(X,\mathbb{Z}(p))$ *fits into the exact sequence*

$$H^{2p-1}(X,\mathbb{Z}) \longrightarrow \mathbb{H}^{2p-1}\big(X,\Omega_X^{\bullet\leqslant p-1}\big) \longrightarrow H^{2p}_{\mathrm{Del}}(X,\mathbb{Z}(p)) \longrightarrow H^{2p}(X,\mathbb{Z}).$$

(ii) *The complex* $\mathbb{Z}_{D,X}(p)[1]$ *is quasi-isomorphic to the cone of the morphism*

$$\mathbb{Z}_X \oplus \Omega_X^{\bullet\geqslant p} \xrightarrow{(2i\pi)^p,\, i} \Omega_X^{\bullet}\ .$$

Thus we have a long exact sequence:

$$\cdots \longrightarrow H^{k-1}(X,\mathbb{C}) \longrightarrow H^k_{\mathrm{Del}}(X,\mathbb{Z}(p)) \longrightarrow \mathbb{H}^k\big(X,\Omega_X^{\bullet\geqslant p}\big) \oplus H^k(X,\mathbb{Z}) \longrightarrow H^k(X,\mathbb{C}) \longrightarrow \cdots$$

and a similar exact sequence can be written with support in a closed subset Z.

(iii) *A cup-product*

$$H^i_{\mathrm{Del}}(X,\mathbb{Z}(p)) \otimes_{\mathbb{Z}} H^j_{\mathrm{Del}}(X,\mathbb{Z}(q)) \longrightarrow H^{i+j}_{\mathrm{Del}}(X,\mathbb{Z}(p+q))$$

is defined and endows $H^*_{\mathrm{Del}}(X)$ *with a ring structure.*

(iv) *If* $f\colon X \longrightarrow Y$ *is a holomorphic map between two smooth complex manifolds, we have a pullback morphism* $f^*\colon H^i_{\mathrm{Del}}(Y,\mathbb{Z}(p)) \longrightarrow H^i_{\mathrm{Del}}(X,\mathbb{Z}(p))$ *which is a ring morphism.*

(v) *If* X *is smooth, compact, and if* E *is a holomorphic vector bundle on* X *of rank* r*, then* $H^*_{\mathrm{Del}}(\mathbb{P}(E))$ *is a free* $H^*_{\mathrm{Del}}(X)$*-module with basis* $1,\ c_1(\mathcal{O}_E(1)),\dots,c_1(\mathcal{O}_E(1))^{r-1}$.

(vi) *For every* t *in* $\mathbb{P}^1$*, let* j_t *be the inclusion* $X \simeq X\times\{t\} \hookrightarrow X\times\mathbb{P}^1$ *. Then the pullback morphism* $j_t^*\colon H^*_{\mathrm{Del}}(X\times\mathbb{P}^1) \longrightarrow H^*_{\mathrm{Del}}(X)$ *is independent of* t *(homotopy principle).*

The assertions (i) and (ii) are straightforward. The cup product in (iii) comes from a morphism of complexes $\mathbb{Z}(p)\otimes_{\mathbb{Z}}\mathbb{Z}(q) \longrightarrow \mathbb{Z}(p+q)$, see [**Es-Vi**, § 1]. This morphism if functorial with respect to pullbacks, which gives (iv). Property (v) is proved by dévissage using the exact sequence

$$0 \longrightarrow \Omega_X^{p-1}[-p] \longrightarrow \mathbb{Z}_{D,X}(p) \longrightarrow \mathbb{Z}_{D,X}(p-1) \longrightarrow 0$$

and the five lemma (see [**Es-Vi**, § 8]). Property (vi) is a consequence of (v): if α is a Deligne class in $H^*_{\mathrm{Del}}(X\times\mathbb{P}^1)$, we can write $\alpha = \mathrm{pr}_1^*\lambda + \mathrm{pr}_1^*\mu\,.\,c_1\big(\mathrm{pr}_2^*\mathcal{O}_{\mathbb{P}^1}(1)\big)$. Thus $j_t^*\alpha = \lambda$.

Remark that (vi) is false if we replace $\mathbb{P}^1(\mathbb{C})$ by $\mathbb{C}$, in contrast with the algebraic case. Indeed, take an elliptic curve S and choose an isomorphism $\phi\colon S \longrightarrow \mathrm{Pic}^0(S)$. There is a universal line bundle $\mathcal{L}$ on $S\times S$ such that for all x in S, $\mathcal{L}_{|S\times x} \simeq \phi(x)$. Let $\pi\colon \mathbb{C} \longrightarrow S$ be the universal covering map of S. Consider the class $\alpha = c_1\big[(\mathrm{id},\pi)^*\mathcal{L}\big]$, then for all t in $\mathbb{C}$, $j_t^*\alpha = c_1\big[\phi(\pi(t))\big]$.

We will now consider more refined properties of Deligne cohomology.

Proposition 3.4 (see [**EZZ**, § 2]).

(i) *If X is a smooth complex manifold and Z is a smooth submanifold of X of codimension d, there exists a cycle class $\{Z\}_D$ in $H^{2d}_{\mathrm{Del},Z}(X,\mathbb{Z}(d))$ compatible with the Bloch cycle class (see* [**Bl**, § 5], [**Es-Vi**, § 6]*) and the topological cycle class. If Z and Z' intersect transversally, $\{Z\cap Z'\}_D = \{Z\}_D\,.\,\{Z'\}_D$. If Z is a smooth hypersurface of X, the image of $\{Z\}_D$ in $H^2_D(X,\mathbb{Z}(1))\simeq \mathrm{Pic}(X)$ is the class of $\mathcal{O}_X(Z)$.*

(ii) *More generally, let $f\colon X\longrightarrow Y$ be a proper holomorphic map between smooth complex manifolds and $d=\dim Y-\dim X$. Then there exists a Gysin morphism*

$$f_*\colon H^{2p}_{\mathrm{Del}}(X,\mathbb{Z}(q))\longrightarrow H^{2(p+d)}_{\mathrm{Del}}(Y,\mathbb{Z}(q+d))$$

compatible with the usual Gysin morphisms in integer and analytic de Rham cohomology. If Z is a smooth submanifold of codimension d of X and $i_Z : Z\longrightarrow X$ is the canonical inclusion, then $i_{Z*}(1)$ is the image of $\{Z\}_D$ in $H^{2d}_{\mathrm{Del}}(X,\mathbb{Z}(d))$.

The detailed proof of this proposition can be found in Chapter 3, Sections 4, 5, 6 and 7. We give here the outline of the proof.

The point (i) is easy to understand. By Proposition 3.3 (ii), since $H^{2d-1}_Z(X,\mathbb{Z})=0$, we have an exact sequence

$$0\longrightarrow H^{2d}_{\mathrm{Del},Z}(X,\mathbb{Z}(d))\longrightarrow H^{2d}_Z(X,\Omega_X^{\bullet\geqslant d})\oplus H^{2d}_Z(X,\mathbb{Z})\longrightarrow H^{2d}_Z(X,\mathbb{C}).$$

The couple $\big((2i\pi)^d\{Z\}_{\mathrm{Bloch}},\{Z\}_{\mathrm{top}}\big)$ is mapped to 0 in $H^{2d}_Z(X,\mathbb{C})$ (see [**Es-Vi**, § 7]). Therefore, it defines a unique element $\{Z\}_D$ in $H^{2d}_{\mathrm{Del},Z}(X,\mathbb{Z}(d))$.

For (ii), we introduce the sheaves $\mathcal{D}^k_{X,\mathbb{Z}}$ of locally integral currents of degree k as done in [**EZZ**, § 2], and [**Gi-So**, § 2.2]. These sheaves are, in a way to be properly defined, a completion of the currents induced by smooth integral chains on X (see [**Ki**, § 2.1] and [**Fe**, § 4.1.24]). Then

- $\mathcal{D}^\bullet_{X,\mathbb{Z}}$ is a soft resolution of $\mathbb{Z}_X$.
- $\mathcal{D}^k_{X,\mathbb{Z}}$ is a subsheaf of $\mathcal{D}^k_X$ stable by push-forward under proper C^∞ maps, where $\mathcal{D}^k_X$ is the sheaf of usual currents of degree k on X.

Thus $\mathbb{Z}_{D,X}(p)[1]$ is quasi-isomorphic to the cone of the morphism

$$\big([2i\pi]^p,i\big)\colon\mathcal{D}^\bullet_{X,\mathbb{Z}}\oplus F^p\mathcal{D}^\bullet_X\longrightarrow\mathcal{D}^\bullet_X.$$

We will denote by $\widetilde{\mathbb{Z}}_{D,X}(p)$ this cone shifted by minus one. Since the sheaves $\mathcal{D}^k_{X,\mathbb{Z}}$, $F^p\mathcal{D}^k_X$ and $\mathcal{D}^k_X$ are acyclic, $Rf_*\mathbb{Z}_{D,X}(p)[1]$ is quasi-isomorphic to the cone of the morphism

$$\big([2i\pi]^p,i\big)\colon f_*\mathcal{D}^\bullet_{X,\mathbb{Z}}\oplus f_*F^p\mathcal{D}^\bullet_X\longrightarrow f_*\mathcal{D}^\bullet_X.$$

The push-forward of currents by f gives an explicit morphism

$$f_*\colon f_*\widetilde{\mathbb{Z}}_{D,X}(p)\longrightarrow\widetilde{\mathbb{Z}}_{D,Y}(p+d)[2d]$$

and then a morphism

$$f_*\colon Rf_*\mathbb{Z}_{D,X}(p)\longrightarrow\mathbb{Z}_{D,Y}(p+d)[2d]$$

in the derived category $\mathcal{D}^b\big(\mathrm{Mod}(\mathbb{Z}_Y)\big)$. We get the Gysin morphism by taking the hypercohomology on Y.

We now state all the properties of the Gysin morphism needed here. The points (vi) and (vii) use Chern classes of vector bundles. These classes are defined in the next section (Section 3.2) and their construction uses only Proposition 3.3 (v) which has been already proved.

PROPOSITION 3.5.

(i) f_* *is compatible with the composition of maps and satisfies the projection formula.*

$$f_*\big(x\,.\,f^*y\big) = f_*x\,.\,y.$$

In particular, if $\Gamma_f \subseteq X \times Y$ *is the graph of* f *and if* X *is compact, then for every Deligne class* α *in* X*,*

$$f_*\alpha = p_{2*}\Big(p_1^*\alpha\,.\,\{\Gamma_f\}_D\Big).$$

(ii) *Consider the cartesian diagram*

$$\begin{tikzcd} Y \times Z \arrow[r, hook, "i_{Y\times Z}"] \arrow[d, "p"'] & X \times Z \arrow[d, "q"] \\ Y \arrow[r, hook, "i_Y"'] & X \end{tikzcd}$$

Then $q^* i_{Y*} = i_{Y\times Z*}\, p^*$.

(iii) *If* $f\colon X \longrightarrow Y$ *is proper and generically finite of degree* d*, then* $f_*f^* = d \times \mathrm{id}$.

(iv) *Consider the cartesian diagram, where* Y *and* Z *are compact and intersect transversally:*

$$\begin{tikzcd} W \arrow[r, hook, "i_{W\to Y}"] \arrow[d, hook, "i_{W\to Z}"'] & Y \arrow[d, hook, "i_Y"] \\ Z \arrow[r, hook, "i_Z"'] & X \end{tikzcd}$$

Then $i_Y^*\, i_{Z*} = i_{W\longrightarrow Y*}\, i^*_{W\longrightarrow Z}$.

(v) *Let* $f\colon X \longrightarrow Y$ *be a surjective map between smooth complex compact manifolds, and let* D *be a smooth hypersurface of* Y *such that* $f^{-1}(D)$ *is a simple normal crossing divisor. Let us write* $f^*D = m_1\widetilde{D}_1 + \cdots + m_N\widetilde{D}_N$*. Let* $\overline{f}_i\colon \widetilde{D}_i \longrightarrow D$ *be the restriction of* f *to* $\widetilde{D}_i$*. Then*

$$f^*\, i_{D*} = \sum_{i=1}^{N} m_i\, i_{\widetilde{D}_i *}\overline{f}_i{}^*.$$

(vi) *Let* X *be compact, smooth, and let* Y *be a smooth submanifold of codimension* d *of* X*. Let* $\widetilde{X}$ *be the blowup of* X *along* Y*, as shown in the following diagram:*

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "q"'] & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

Then we have an isomorphism

$$\begin{aligned} H^*_{\mathrm{Del}}(X) \oplus \bigoplus_{i=1}^{d-1} H^*_{\mathrm{Del}}(Y) &\longrightarrow H^*_{\mathrm{Del}}(\widetilde{X}) \\ \big(x, (y_i)_{1\leq i\leq d-1}\big) &\longmapsto p^*x + \sum_{i=1}^{d-1} j_*\Big[y_i\, c_1\big(\mathcal{O}_{N_{Y/X}}(-1)\big)^{i-1}\Big]. \end{aligned}$$

In particular, if α is a Deligne class on $\widetilde{X}$ such that $j^*\alpha$ is the pullback of a Deligne class on Y, then α is the pullback of a unique Deligne class on X.

Moreover, if F is the excess conormal bundle defined by the exact sequence

$$0 \longrightarrow F \longrightarrow q^*N^*_{Y/X} \longrightarrow N^*_{E/\widetilde{X}} \longrightarrow 0,$$

we have the excess formula $p^*\, i_*\, \alpha = j_*\big(q^*\alpha\; c_{d-1}(F^*)\big)$.

(vii) *If Y is a smooth compact submanifold of X of codimension d, we have the auto-intersection formula*

$$i_Y^*\, i_{Y*}\alpha = \alpha\, c_d\big(N_{Z/X}\big).$$

Proof. Property (i) is a consequence of the functoriality of push-forward for currents. For the projection formula, we use the complex $\widetilde{\mathbb{Z}}_{D,X}(\,.\,)$ for the variable x and the complex $\mathbb{Z}_{D,Y}(\,.\,)$ for the variable y. To prove the last formula of (i), we remark that p_2 is proper since X is compact. Then we write

$$\begin{aligned} f_*\alpha = p_{2*}(\mathrm{id}, f)_*\alpha &= p_{2*}(\mathrm{id}, f)_*\big[(\mathrm{id}, f)^*p_1^*\alpha\big] \\ &= p_{2*}\big(p_1^*\alpha\,.\,(\mathrm{id}, f)_*(1)\big) = p_{2*}\big(p_1^*\alpha\,.\,\{\Gamma_f\}_D\big). \end{aligned}$$

For (ii), we can pull-back currents under p and q since these morphisms are submersions. Then p^* and q^* are defined for the complexes $\widetilde{\mathbb{Z}}_D(\,.\,)$ and (ii) is an equality of complexes morphisms.

The detailed proofs of (i) and (ii) can be found in Chapter 3, Proposition 7.1, page 122.

For (iii), it is enough by the projection formula to prove that $f_*(1) = d$, which is well known.

Let us prove (iv). Let Γ_Z (resp. $\Gamma_{W/Y}$, resp. $\Gamma_{W/Z}$) be the graphs of i_Z (resp. $i_{W\to Y}$, resp. $i_{W\to Z}$) in $Z\times X$ (resp. $W\times Y$, resp. $W\times Z$), and $\Gamma = \big(i_{W\to Z}, \mathrm{id}\big)_*\Gamma_{W/Y} \subseteq Z\times Y$. Since Z and Y meet transversally in X, $\{\Gamma\}_D = \big(\mathrm{id}, i_Y\big)^*\{\Gamma_Z\}_D$. Let p_1 and p_2 (resp. p_1' and p_2', resp. p_1'' and p_2'') be the projections of $Z\times X$ (resp. $Z\times Y$, resp. $W\times Y$) on the first and second factor. Then

$$\begin{aligned} i_Y^*\, i_{Z*}\,\alpha &= i_Y^*\, p_{2*}\big(p_1^*\alpha\,.\,\{\Gamma_Z\}_D\big) && \text{by (i)} \\ &= p'_{2*}(\mathrm{id}, i_Y)^*\big(p_1^*\alpha\,.\,\{\Gamma_Z\}_D\big) && \text{by (ii)} \\ &= p'_{2*}\Big(p_1'^{\,*}\alpha\,.\,\big[(\mathrm{id}, i_Y)^*\{\Gamma_Z\}_D\big]\Big) && \text{by the projection formula} \\ &= p'_{2*}\big(p_1'^{\,*}\alpha\,.\,\{\Gamma\}_D\big) \\ &= p'_{2*}\Big(p_1'^{\,*}\alpha\,.\,\big(i_{W\to Z}, \mathrm{id}\big)_*\{\Gamma_{W/Y}\}_D\Big) \\ &= p''_{2*}\Big(\big(i_{W\to Z}, \mathrm{id}\big)^*p_1'^{\,*}\,\alpha\,.\,\{\Gamma_{W/Y}\}_D\Big) && \text{by the projection formula} \\ &= p''_{2*}\big(p_1''^{\,*}\, i^*_{W\to Z}\,\alpha\,.\,\{\Gamma_{W/Y}\}_D\big) \\ &= i_{W\to Y*}\, i^*_{W\to Z}\,\alpha \end{aligned}$$

We now prove (v). We will define first some notations: Let Γ be the graph of $i_D : D \hookrightarrow Y$ and Γ_i be the graph of $i_{\widetilde{D}_i} : \widetilde{D}_i \hookrightarrow X$. We define $\Gamma'_i = \big(\overline{f}_i, \mathrm{id}\big)_*\big(\Gamma_i\big) \subseteq D \times X$. We call $p_1 : D \times Y \longrightarrow D$ and $p_2 : D \times Y \longrightarrow Y$ the first and second projections. In the same manner, we define the projections $p'_1 : D \times X \longrightarrow D$, $p'_2 : D \times X \longrightarrow X$, $p'_{1,i} : \widetilde{D}_i \times X \longrightarrow \widetilde{D}_i$, and $p'_{2,i} : \widetilde{D}_i \times X \longrightarrow X$.

We have $(\mathrm{id}, f)^*\{\Gamma\}_D = \displaystyle\sum_{i=1}^{N} m_i \{\Gamma'_i\}_D$. Indeed, in local coordinates, we can write

$$\begin{aligned} D &= \big\{(y_1, \dots, y_n) \text{ such that } y_1 = 0\big\} \\ \Gamma_D &= \big\{(y_2, \dots, y_n, y'_1, \dots, y'_n) \text{ such that } y'_1 = 0 \text{ and } y_i = y'_i \text{ for } i \geq 2\big\}. \end{aligned}$$

Now

$$\{\Gamma\}_{\mathrm{Bl}} = \frac{dy'_1}{y'_1} \wedge \frac{d(y_2 - y'_2)}{y_2 - y'_2} \wedge \dots \wedge \frac{d(y_n - y'_n)}{y_n - y'_n} \qquad \text{(see Chapter 3, page 110).}$$

If $f = (f_1, \dots, f_n)$, there exist local coordinates $x_1, \dots, x_N$ on X such that $f_1 = x_1^{m_1} \dots x_N^{m_N}$. Thus

$$\begin{aligned} (\mathrm{id}, f)^*\{\Gamma\}_{\mathrm{Bl}} &= \frac{df_1}{f_1} \wedge \frac{d(y_2 - f_2)}{y_2 - f_2} \wedge \dots \wedge \frac{d(y_n - f_n)}{y_n - f_n} \\ &= \sum_{i=1}^{N} m_i \frac{dx_i}{x_i} \wedge \frac{d(y_2 - f_2)}{y_2 - f_2} \wedge \dots \wedge \frac{d(y_n - f_n)}{y_n - f_n}. \end{aligned}$$

In $D \times X$ we have $\Gamma'_i = \big\{(y_2, \dots, y_m, x) \text{ such that } x_i = 0,\ y_j = f_j(x) \text{ for } j \geq 2\big\}$. This gives $(\mathrm{id}, f)^*\{\Gamma\}_{\mathrm{Bl}} = \displaystyle\sum_{i=1}^{N} m_i \{\Gamma'_i\}_{\mathrm{Bl}}$ and also $(\mathrm{id}, f)^*\{\Gamma\}_{\mathrm{Del}} = \displaystyle\sum_{i=1}^{N} m_i \{\Gamma'_i\}_{\mathrm{Del}}$ since the Deligne cycle class is determined by the Bloch cycle class.

Then

$$\begin{aligned} f^* i_{D*} \alpha &= f^* p_{2*}\big(p_1^*\alpha \,.\, \{\Gamma\}_D\big) && \text{by (i)} \\ &= p'_{2*}(\mathrm{id}, f)^*\big(p_1^*\alpha \,.\, \{\Gamma\}_D\big) && \text{by (ii)} \\ &= \sum_{i=1}^{N} m_i\, p'_{2*}\big({p'_1}^*\alpha \,.\, \{\Gamma'_i\}_D\big) && \text{by the projection formula} \\ &= \sum_{i=1}^{N} m_i\, p'_{2*}\big({p'_1}^*\alpha \,.\, (\overline{f}_i, \mathrm{id})_*\{\Gamma_i\}_D\big) \\ &= \sum_{i=1}^{N} m_i\, p'_{2,i*}\big({p'_{1,i}}^*\overline{f}_i^{\,*}\alpha \,.\, \{\Gamma_i\}_D\big) && \text{by the projection formula} \\ &= \sum_{i=1}^{N} m_i\, i_{\widetilde{D}_i*}\, \overline{f}_i^{\,*}\, \alpha. \end{aligned}$$

Before dealing with (vi), we prove (vii) when Y is a hypersurface. In the case of the étale cohomology, it is possible to assume that $\alpha = 1$ (see [**SGA 5**, Exposé VII, § 4] and [**SGA 4$\frac{1}{2}$**, Cycle § 1.2]). Remark that this is no longer possible here, for there is no purity theorem in Deligne cohomology.

We use the deformation to the normal cone, an idea which goes back to Mumford (see [**LMS**] and [**SGA 5**, Exposé VII § 9]). The aim was originally to prove the same formula in the Chow groups. Let $M_{Y/X}$ be the blowup of $X \times \mathbb{P}^1$ along $Y \times \{0\}$, $\widetilde{X} \simeq X$ be the blowup of X along Y, and $M^\circ_{Y/X} = M_{Y/X} \backslash \widetilde{X}$. Then we have an injection $F: Y \times \mathbb{P}^1 \hookrightarrow M^0_{Y/X}$ over $\mathbb{P}^1$ (see [**Fu**, Ch. 5] § 5.1). We denote the inclusions $N_{Y/X} \hookrightarrow M^\circ_{Y/X}$ and $Y \hookrightarrow N_{Y/X}$ by j_0 and i, the projections of $\big(Y \times \mathbb{P}^1\big) \times M^\circ_{Y/X}$ (resp. $Y \times \mathbb{P}^1$, resp. $\big(Y \times \mathbb{P}^1\big) \times N_{Y/X}$, resp. $Y \times N_{Y/X}$) on its first and second factor by pr_1 and pr_2 (resp $\widetilde{\mathrm{pr}}_1$ and $\widetilde{\mathrm{pr}}_2$, resp. pr'_1 and pr'_2, resp. pr''_1 and pr''_2). Besides, $\Gamma \subseteq Y \times \mathbb{P}^1 \times M^\circ_{Y/X}$ is the graph of F, and $\Gamma' \subseteq Y \times N_{Y/X}$ is the graph of i. Finally, $\Gamma'' = \big(i_0, \mathrm{id}_{N_{Y/X}}\big)_* \Gamma' \subseteq Y \times \mathbb{P}^1 \times N_{Y/X}$, where $i_0: Y \times \{0\} \hookrightarrow Y \times \mathbb{P}^1$ is the injection of the central fiber. Remark that pr'_2 and pr''_2 are proper maps since Y is compact.

We have $\big(i_0, \mathrm{id}_{N_{Y/X}}\big)_* \{\Gamma'\}_D = \{\Gamma''\}_D$ and $\big(\mathrm{id}_{Y \times \mathbb{P}^1}, j_0\big)^* \{\Gamma\}_D = \{\Gamma''\}_D$. Let $\gamma = F_*\big(\widetilde{\mathrm{pr}}_1^* \alpha\big)$. Then

$$\begin{aligned}
j_0^* \gamma &= j_0^* \, \mathrm{pr}_{2*}\big(\mathrm{pr}_1^* \widetilde{\mathrm{pr}}_1^* \alpha \,.\, \{\Gamma\}_D\big) = \mathrm{pr}'_{2*}\Big[\big(\mathrm{id}_{Y \times \mathbb{P}^1}, j_0\big)^* \big(\mathrm{pr}_1^* \widetilde{\mathrm{pr}}_1^* \alpha \,.\, \{\Gamma\}_D\big)\Big] && \text{by (ii)} \\
&= \mathrm{pr}'_{2*}\Big[\big(\mathrm{id}_{Y \times \mathbb{P}^1}, j_0\big)^* \, \mathrm{pr}_1^* \widetilde{\mathrm{pr}}_1^* \alpha \,.\, \{\Gamma''\}_D\Big] \\
&= \mathrm{pr}'_{2*}\big(i_0, \mathrm{id}_{N_{Y/X}}\big)_* \Big(\big(i_0, \mathrm{id}_{N_{Y/X}}\big)^* \, \big(\mathrm{id}_{Y \times \mathbb{P}^1}, j_0\big)^* \, \mathrm{pr}_1^* \widetilde{\mathrm{pr}}_1^* \alpha \,.\, \{\Gamma'\}_D\Big) \\
& && \text{(by the projection formula)} \\
&= \mathrm{pr}''_{2*}\big(\mathrm{pr}''^*_1 \alpha \,.\, \{\Gamma'\}_D\big) = i_* \alpha.
\end{aligned}$$

By the homotopy principle (Proposition 3.3 (vi)), the class $F^* \gamma_{|Y \times \{t\}}$ is independent of t. If $t \neq 0$, we have clearly $F^* \gamma_{|Y \times \{t\}} = i_Y^* i_{Y*} \alpha$. For $t = 0$, $F^* \gamma_{|Y \times \{0\}} = i^* j_0^* \gamma = i^* i_* \alpha$. Let $\pi: N_{Y/X} \longrightarrow Y$ be the projection of $N_{Y/X}$ on Y. Then $\alpha = i^* \pi^* \alpha$. Thus

$$i^* i_* \alpha = i^* i_* (i^* \pi^* \alpha) = i^* \big(\pi^* \alpha \,.\, \overline{\{Y\}}_D\big) = \alpha \,.\, i^* \overline{\{Y\}}_D,$$

where $\overline{\{Y\}}_D$ is the cycle class of Y in $N_{Y/X}$.

Now $\overline{\{Y\}}_D = c_1\big(\mathcal{O}_{N_{Y/X}}(Y)\big)$, so that $i^* \overline{\{Y\}}_D = c_1\big(N_{Y/N_{Y/X}}\big) = c_1\big(N_{Y/X}\big)$.

We can now prove (vi). Its first part is straightforward using dévissage as in Proposition 3.3 (v) and the analogous result in Dolbeault cohomology and in integer cohomology.

If α is a Deligne class on $\widetilde{X}$, we can write $\alpha = p^* x + \sum_{i=1}^{d-1} j_* \Big[y_i \, c_1\big(\mathcal{O}_{N_{Y/X}}(-1)\big)^{i-1}\Big]$. Since E is a hypersurface of $\widetilde{X}$, by the formula proved above

$$j^* j_* \, \lambda = \lambda \, c_1\big(N_{E/\widetilde{X}}\big) = \lambda \; c_1\big(\mathcal{O}_{N_{Y/X}}(-1)\big)$$

for any Deligne class λ on E. We obtain

$$j^* \alpha = q^* i^* x + \sum_{i=1}^{d-1} (-1)^i y_i \, c_1\big(\mathcal{O}_{N_{Y/X}}(1)\big)^i.$$

Since $j^* \alpha = q^* \delta$, all the classes y_i vanish by Proposition 3.3 (v). Thus $\alpha = p^* x$. By Proposition 3.5 (iii), $x = p_* \alpha$.

For the excess formula, let α be a Deligne class on Y. We define $\beta = j_*\big(q^*\alpha\ c_{d-1}(F^*)\big)$. Then, by Proposition 3.7 (i) and (ii),

$$j^*\beta = \big[q^*\alpha\ c_{d-1}(F^*)\big]\, c_1\big(N_{E/\widetilde{X}}\big) = q^*\big[\alpha\, c_d\big(N_{Y/X}\big)\big].$$

By the discussion above, β comes from the base, so that

$$\beta = p^*p_*\beta = p^*i_*q_*\big(q^*\alpha\, c_{d-1}(F^*)\big) = p^*i_*\big[\alpha\ q_*\big(c_{d-1}(F^*)\big)\big] = p^*i_*\alpha$$

for $q_*\big(c_{d-1}(F^*)\big) = 1$ (see [**Bo-Se**, Lemme 19.b]).

— Proof of (vii). The formula is already true for $d = 1$. We blowup X along Y, use the excess formula (vi) and we obtain:

$$\begin{aligned} q^*i_Y^*i_{Y*}\,\alpha &= j^*p^*i_{Y*}\,\alpha = j^*\big[j_*\big(q^*\alpha\, c_{d-1}(F^*)\big)\big] \\ &= q^*\alpha\ c_{d-1}(F^*)\, c_1\big(N_{E/\widetilde{X}}\big) = q^*\big(\alpha\, c_d\big(N_{Y/X}\big)\big). \end{aligned}$$

Since q^* is injective, we get the result. ☐

3.2. Chern classes for holomorphic vector bundles. We refer to [**Zu**, § 4] and [**Es-Vi**, § 8] for all this section. From now on, we will suppose that X is smooth of dimension n. Let E be a holomorphic bundle on X of rank r, and $\mathbb{P}(E)$ be the projective bundle of E endowed with the line bundle $\mathcal{O}_E(1)$. Let $\alpha = c_1\big(\mathcal{O}_E(1)\big)$. By property (v) of Proposition 3.3, we can define $\big(c_i(E)\big)_{1\le i\le r}$ by the relation

$$\alpha^r + p^*c_1(E)\,\alpha^{r-1} + \cdots + p^*c_r(E) = 0 \quad \text{in } H^{2r}_{\mathrm{Del}}\big(\mathbb{P}(E), \mathbb{Z}(r)\big).$$

Thus $c_i(E)$ is an element of $H^{2i}_{\mathrm{Del}}(X, \mathbb{Z}(i))$. The knowledge of the Chern classes $c_i(E)$ allows to construct exponential Chern classes $\mathrm{ch}_i(E)$, $0 \le i \le n$, in $H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i))$. These classes are obtained as the values of certain universal polynomials with rational coefficients on $c_1(E), \dots, c_r(E)$ (see [**Hirz**]). They can also be constructed with the splitting principle using projective towers (see [**Gro 1**]). They are completely characterized by the following facts:

- they satisfy the Whitney additivity formula (Proposition 3.7 (i));
- they satisfy the functoriality formula under pullbacks (Proposition 3.7 (ii));
- if L is a line bundle, $\mathrm{ch}(L) = e^{c_1(L)}$ (see Definition 3.6).

In the same spirit, the Todd classes $\mathrm{td}_i(E)$, $0 \le i \le n$, are defined in $H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i))$ and characterized by the following properties

- they satisfy the Whitney multiplicativity formula (Proposition 3.7 (i));
- they are functorial under pullback by holomorphic maps (Proposition 3.7 (ii));
- if L is a line bundle, $\mathrm{td}(L) = \frac{c_1(L)}{1-e^{-c_1(L)}}$ (see Definition 3.6).

Definition 3.6. The total Chern (resp. Todd) class of E is the element $c(E)$ (resp $\mathrm{td}(E)$) of $H^*_{\mathrm{Del}}(X)$ defined by

$$c(E) = 1 + c_1(E) + \cdots + c_r(E), \quad (\text{resp. } \mathrm{td}(E) = td_0(E) + \cdots + \mathrm{td}_n(E)).$$

The Chern character of E is the element $\mathrm{ch}(E)$ of $H^*_{\mathrm{Del}}(X, \mathbb{Q})$ defined by

$$\mathrm{ch}(E) = \mathrm{ch}_0(E) + \cdots + \mathrm{ch}_n(E).$$

The splitting machinery gives the following proposition:

Proposition 3.7.

(i) *If* $0 \longrightarrow E \longrightarrow F \longrightarrow G \longrightarrow 0$ *is an exact sequence of vector bundles on X, the Whitney formula holds:* $c(F) = c(E)c(G)$, $\mathrm{td}(F) = \mathrm{td}(E)\,\mathrm{td}(G)$ *and* $\mathrm{ch}(F) = \mathrm{ch}(E) + \mathrm{ch}(G)$.

(ii) *If f is a holomorphic map between X and Y and if E is a holomorphic bundle on Y, we have $c\big(f^*E\big) = f^*c(E)$ and $\mathrm{ch}\big(f^*E\big) = f^*\,\mathrm{ch}(E)$.*
(iii) *If E and F are two holomorphic vector bundles on X, $\mathrm{ch}(E\otimes F) = \mathrm{ch}(E)\,\mathrm{ch}(F)$.*

NOTATION 3.8. From now on, if $\mathcal{E}$ is a locally free sheaf and E is the associated holomorphic vector bundle, we will denote by $\overline{\mathrm{ch}}(\mathcal{E})$ the Chern character $\mathrm{ch}(E)$. Thus $\overline{\mathrm{ch}}$ is well defined on a basis of any dimension but only for locally free sheaves, and we will make use of it in our construction.

4. Construction of Chern classes

The construction of exponential Chern classes $\mathrm{ch}_p(\mathcal{F})$ in $H^{2p}_{\mathrm{Del}}(X, \mathbb{Q}(p))$ for an arbitrary coherent sheaf $\mathcal{F}$ on X will be done by induction on $\dim X$. If $\dim X = 0$, X is a point and everything is obvious.

Let us now precisely state the induction hypotheses (H_n):

(W_n) If $\dim X \leq n$ and $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is an exact sequence of analytic sheaves on X, then $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$. This means that ch is defined on $K(X)$ and is a group morphism.

(F_n) If $\dim X \leq n$, $\dim Y \leq n$, and if $f\colon X \longrightarrow Y$ is a holomorphic map, then for all y in $K(Y)$, $\mathrm{ch}(f^{\dagger}y) = f^*\,\mathrm{ch}(y)$.

(C_n) If $\dim X \leq n$, the Chern classes are compatible with those constructed in Part 3 on locally free sheaves, i.e. $\mathrm{ch}(\mathcal{F}) = \overline{\mathrm{ch}}(\mathcal{F})$ for every locally free sheaf $\mathcal{F}$.

(P_n) If $\dim X \leq n$, ch is a ring morphism: if x, y are two elements of $K(X)$, then $\mathrm{ch}(x\,.\,y) = \mathrm{ch}(x)\,\mathrm{ch}(y)$ and $\mathrm{ch}(1) = 1$.

(RR_n) If Z is a smooth hypersurface of X, where $\dim X \leq n$, then the (GRR) theorem holds for i_Z: for every coherent sheaf $\mathcal{F}$ on Z, $\mathrm{ch}(i_{Z*}\mathcal{F}) = i_{Z*}\big(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1}\big)$.

For the definition of analytic K-theory and related operations we refer to [**Bo-Se**].
From now on, we will suppose that all the properties of the induction hypotheses (H_{n-1}) above are true.

THEOREM 4.1. *Assuming hypotheses (H_{n-1}), we can define a Chern character for analytic coherent sheaves on compact complex manifolds of dimension n. Furthermore it satisfies (P_n), (F_n), (RR_n), (W_n) and (C_n).*

Let us briefly explain the organization of the proof of this theorem. In § 4.1, we construct the Chern character for torsion sheaves. In § 4.3, we construct the Chern character for arbitrary coherent sheaves, using the results of § 4.2. Properties (RR_n) for a smooth hypersurface and (C_n) will be obvious consequences of the construction. In § 5.3, we prove (W_n) and then (F_n) and (P_n) using the preliminary results of § 5.1 and § 5.2. Finally, we prove (RR_n) in § 6.

4.1. Construction for torsion sheaves. In this section, we define Chern classes for torsion sheaves by forcing the Grothendieck-Riemann-Roch formula for immersions of smooth hypersurfaces. Let $K_{\mathrm{tors}}(X)$ denote the Grothendieck group of the abelian category of torsion sheaves on X. We will prove the following version of Theorem 4.1 for torsion sheaves:

PROPOSITION 4.2. *On any n-dimensional complex manifold, we can define a Chern character for torsion sheaves such that:*

(i) (W_n) *If $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is an exact sequence of torsion sheaves on X with $\dim X \leq n$, then $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$. This means that* ch *is a group morphism defined on $K_{\mathrm{tors}}(X)$.*

(ii) (P_n) *Let $\mathcal{E}$ be a locally free sheaf and x be an element of $K_{\mathrm{tors}}(X)$. Then*

$$\mathrm{ch}([\mathcal{E}].\,x) = \overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}(x).$$

(iii) (F_n) *Let $f\colon X \longrightarrow Y$ be a holomorphic map where $\dim X \leq n$ and $\dim Y \leq n$, and $\mathcal{F}$ be a coherent sheaf on Y such that $\mathcal{F}$ and $f^*\mathcal{F}$ are torsion sheaves. Then*

$$\mathrm{ch}(f^{\dagger}[\mathcal{F}]) = f^*\,\mathrm{ch}(\mathcal{F}).$$

(iv) (RR_n) *If Z is a smooth hypersurface of X and $\mathcal{F}$ is coherent on Z, then*

$$\mathrm{ch}\big(i_{Z*}\mathcal{F}\big) = i_{Z*}\big(\mathrm{ch}(\mathcal{F})\,\mathrm{td}(N_{Z/X})^{-1}\big).$$

We will proceed in three steps. In § 4.1.1, we perform the construction for coherent sheaves supported in a smooth hypersurface. In § 4.1.2, we deal with sheaves supported in a simple normal crossing divisor. In § 4.1.3, we study the general case.

4.1.1. Let Z be a smooth hypersurface of X where $\dim X \leq n$. For $\mathcal{G}$ coherent on Z, we define $\mathrm{ch}\big(i_{Z*}\mathcal{G}\big)$ by the GRR formula $\mathrm{ch}\big(i_{Z*}\mathcal{G}\big) = i_{Z*}\big(\mathrm{ch}(\mathcal{G})\,\mathrm{td}(N_{Z/X})^{-1}\big)$, where $\mathrm{ch}(\mathcal{G})$ is defined by induction.

If $0 \longrightarrow \mathcal{G}' \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}'' \longrightarrow 0$ is an exact sequence of coherent sheaves on Z, by (W_{n-1}), we have $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{G}') + \mathrm{ch}(\mathcal{G}'')$. Thus $\mathrm{ch}\big(i_{Z*}\mathcal{G}\big) = \mathrm{ch}\big(i_{Z*}\mathcal{G}'\big) + \mathrm{ch}\big(i_{Z*}\mathcal{G}''\big)$. We obtain now a well-defined morphism

$$\begin{tikzcd} K(Z) \arrow[r, "\sim", "i_{Z*}"'] & K_Z(X) \arrow[d, "\mathrm{ch}_Z"] \\ & H^*_{\mathrm{Del}}(X, \mathbb{Q}) \end{tikzcd}$$

Remark that if $\mathcal{G}$ is a coherent sheaf on X which can be written $i_{Z*}\mathcal{F}$, then the hypersurface Z is not necessarily unique. If Z is chosen, $\mathcal{F}$ is of course unique. This is the reason why we use the notation $\mathrm{ch}_Z(\mathcal{G})$. We will see in Proposition 4.7 that $\mathrm{ch}_Z(\mathcal{G})$ is in fact independent of Z.

The assertions of the following proposition are particular cases of (C_n), (F_n), and (P_n).

Proposition 4.3. *Let Z be a smooth hypersurface of X.*

(i) *For all x in $K_Z(X)$, $\mathrm{ch}\big(i_Z^{\dagger}x\big) = i_Z^{\dagger}\,\mathrm{ch}_Z(x)$.*

(ii) *If $\mathcal{E}$ is a locally free sheaf on X and x is an element of $K_Z(X)$, then*

$$\mathrm{ch}_Z([\mathcal{E}].\,x) = \overline{\mathrm{ch}}(\mathcal{E})\,.\,\mathrm{ch}_Z(x).$$

Proof. (i) We have $x = \overline{x}\,\raisebox{0.1ex}{\textbf{.}}_Z\,[i_{Z*}\mathcal{O}_Z]$ in $K_Z(X)$, where $\overline{x}$ is defined in Appendix 7.2. Thus,

$$\begin{aligned} i_Z^*\,\mathrm{ch}_Z(x) &= i_Z^* i_{Z*}\big(\mathrm{ch}(\overline{x})\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\big) \\ &= \mathrm{ch}(\overline{x})\,\mathrm{td}\big(N_{Z/X}\big)^{-1} c_1\big(N_{Z/X}\big) && \text{by Proposition 3.5 (vii)} \\ &= \mathrm{ch}(\overline{x})\Big[1 - e^{-c_1\left(N_{Z/X}\right)}\Big] \\ &= \mathrm{ch}(\overline{x})\,\mathrm{ch}\big(i_Z^{\dagger} i_{Z*}\mathcal{O}_Z\big) && \text{by Proposition 7.6 (i) and } (\mathrm{C}_{n-1}) \\ &= \mathrm{ch}\big(\overline{x}\,.\,i_Z^{\dagger} i_{Z*}\mathcal{O}_Z\big) && \text{by } (\mathrm{P}_{n-1}) \\ &= \mathrm{ch}\big(i_Z^{\dagger}x\big) && \text{By Proposition 7.5.} \end{aligned}$$

(ii) We have

$$
\begin{aligned}
\mathrm{ch}_Z\big([\mathcal{E}]\,.\,x\big) &= \mathrm{ch}_Z\big(i_{Z*}(i_Z^{\dagger}[\mathcal{E}]\,.\,\overline{x})\big) \\
&= i_{Z*}\Big(\mathrm{ch}\big(i_Z^{\dagger}\,[\mathcal{E}]\,.\,\overline{x}\big)\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\Big) \\
&= i_{Z*}\Big(i_Z^{*}\,\overline{\mathrm{ch}}\big(\mathcal{E}\big)\,\mathrm{ch}\big(\overline{x}\big)\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\Big) \qquad \text{by Proposition 3.7 (ii), } (\mathrm{P}_{n-1}) \text{ and } (\mathrm{C}_{n-1}) \\
&= \overline{\mathrm{ch}}\big(\mathcal{E}\big)\,i_{Z*}\Big(\mathrm{ch}\big(\overline{x}\big)\,\mathrm{td}\big(N_{Z/X}\big)^{-1}\Big) \qquad \text{by the projection formula} \\
&= \overline{\mathrm{ch}}\big(\mathcal{E}\big)\,\mathrm{ch}_Z(x).
\end{aligned}
$$

□

4.1.2. Let D be a divisor in X with simple normal crossing. By Proposition 7.8, we have an exact sequence:

$$\bigoplus_{i<j} K_{D_{ij}}(X) \longrightarrow \bigoplus_{i} K_{D_i}(X) \longrightarrow K_D(X) \longrightarrow 0.$$

Let us consider the morphism $\bigoplus_i \mathrm{ch}_{D_i}$. If $\mathcal{F}$ belongs to $K(D_{ij})$, then

$$\mathrm{ch}_{D_i}\big(i_{D_{ij}*}\mathcal{F}\big) = i_{D_i*}\Big(\mathrm{ch}\big(i_{D_{ij}\,\longrightarrow\, D_i*}\mathcal{F}\big)\,\mathrm{td}\big(N_{D_i/X}\big)^{-1}\Big) = i_{D_{ij}*}\Big(\mathrm{ch}(\mathcal{F})\,\mathrm{td}\big(N_{D_{ij}/X}\big)^{-1}\Big)$$

because of (RR_{n-1}) and the multiplicativity of the Todd class.
Thus $\mathrm{ch}_{D_i}\big(i_{D_{ij}*}\mathcal{F}\big) = \mathrm{ch}_{D_j}\big(i_{D_{ij}*}\mathcal{F}\big)$, and we get a map $\mathrm{ch}_D\colon K_D(X) \longrightarrow H^*_{\mathrm{Del}}(X,\mathbb{Q})$ such that the diagram

$$
\begin{tikzcd}
\bigoplus_i K_{D_i}(X) \arrow[r] \arrow[dr, "\bigoplus_i \mathrm{ch}_{D_i}"'] & K_D(X) \arrow[r] \arrow[d, "\mathrm{ch}_D"] & 0 \\
& H^*_{\mathrm{Del}}(X,\mathbb{Q}) &
\end{tikzcd}
$$

is commutative.

Proposition 4.4. *The classes* ch_D *have the following properties:*

(i) *If* $\mathcal{E}$ *is a locally free sheaf on* X *and* x *is an element of* $K_D(X)$*, then*

$$\mathrm{ch}_D([\mathcal{E}]\,.\,x) = \overline{\mathrm{ch}}(\mathcal{E}).\ \mathrm{ch}_D(x).$$

(ii) *Let* $\widetilde{D}$ *be an effective simple normal crossing divisor in* X *such that* $\widetilde{D}^{red} = D$*. Then*

$$\mathrm{ch}_D\big(\mathcal{O}_{\widetilde{D}}\big) = 1 - \overline{\mathrm{ch}}\big(\mathcal{O}_X(-\widetilde{D})\big).$$

(iii) (First lemma of functoriality) *Let* $f\colon X \longrightarrow Y$ *be a surjective map. Let* D *be a reduced divisor in* Y *with simple normal crossing such that* $f^{-1}(D)$ *is also a divisor with simple normal crossing in* X*. Then for all* y *in* $K_D(Y)$*,*

$$\mathrm{ch}_{f^{-1}(D)}\big(f^{\dagger}y\big) = f^*\,\mathrm{ch}_D(y).$$

(iv) (Second lemma of functoriality) *Let* Y *be a smooth submanifold of* X *and* D *be a reduced divisor in* X *with simple normal crossing. Then for all* x *in* $K_D(X)$*,*

$$\mathrm{ch}\big(i_Y^{\dagger}x\big) = i_Y^*\,\mathrm{ch}_D(x).$$

Proof. We start with two technical lemmas which will be crucial for the proof of (ii) and (iii).

LEMMA 4.5. *Let $D = m_1 D_1 + \cdots + m_N D_N$ be an effective simple normal crossing divisor in X, and μ be the element of $H^*_{\mathrm{Del}}(X,\mathbb{Q})$ defined by*

$$\mu = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k!}\Big(m_1\{D_1\}_D + \cdots + m_N\{D_N\}_D\Big)^{k-1}.$$

Then there exist u_i in $K_{D_i}(X)$, $1 \leq i \leq N$, and ζ_{ij} in $H^*_{\mathrm{Del}}(D_{ij})$, $1 \leq i, j \leq N$, $i \neq j$, such that

(a) $u_1 + \cdots + u_N = \mathcal{O}_D$ *in* $K_{D^{red}}(X)$.

(b) $\zeta_{ij} = -\zeta_{ji}$, $1 \leq i, j \leq N$, $i \neq j$.

(c) $\mathrm{ch}(\overline{u}_i)\,\mathrm{td}\big(N_{D_i/X}\big)^{-1} - m_i\, i^*_{D_i}\,\mu = \displaystyle\sum_{\substack{j=1\\ j\neq i}}^{N} i_{D_{ij} \longrightarrow D_i *}\zeta_{ij}, \quad 1 \leq i \leq N.$

PROOF. We proceed by induction on the number N of branches of D^{red}.
If $N = 1$, we must prove that $\mathrm{ch}(\overline{u}_1)\,\mathrm{td}\big(N_{D_1/X}\big)^{-1} = m_1\, i^*_{D_1}\,\mu$, where $u_1 = \mathcal{O}_{m_1 D_1}$. In $K_{D_1}(X)$ we have $\mathcal{O}_{m_1 D_1} = \displaystyle\sum_{q=0}^{m_1-1} i_{D_1 *}\big(N^{*\otimes q}_{D_1/X}\big)$, thus $\overline{u}_1 = \displaystyle\sum_{q=0}^{m_1-1} N^{*\otimes q}_{D_1/X}$. Therefore

$$\begin{aligned}\mathrm{ch}(\overline{u}_1)\,\mathrm{td}\big(N_{D_1/X}\big)^{-1} &= \left(\sum_{q=0}^{m_1-1} e^{-q\,c_1(N_{D_1/X})}\right)\frac{1-e^{-c_1(N_{D_1/X})}}{c_1\big(N_{D_1/X}\big)}\\ &= \frac{1-e^{-m_1 c_1(N_{D_1/X})}}{c_1\big(N_{D_1/X}\big)} = m_1\, i^*_{D_1}\,\mu.\end{aligned}$$

Suppose that the lemma holds for divisors D' such that D'_{red} has $N-1$ branches. Let $D = m_1 D_1 + \cdots + m_N D_N$ and $D' = m_1 D_1 + \cdots + m_{N-1} D_{N-1}$. By induction, there exist u'_i in $K_{D_i}(X)$, $1 \leq i \leq N-1$, and ζ'_{ij} in $H^*_{\mathrm{Del}}(D_{ij})$, $1 \leq i, j \leq N-1$, $i \neq j$, satisfying properties (a), (b), and (c) of Lemma 4.5. For $0 \leq k \leq m_N$, we introduce the divisors $Z_k = m_1 D_1 + \cdots + m_{N-1} D_{N-1} + k D_N$. We have exact sequences

$$0 \longrightarrow i^*_{D_N}\mathcal{O}_X(-Z_k) \longrightarrow \mathcal{O}_{Z_{k+1}} \longrightarrow \mathcal{O}_{Z_k} \longrightarrow 0.$$

Thus, in $K_{D^{\mathrm{red}}}(X)$, we have

$$\mathcal{O}_D = \mathcal{O}_{D'} + i_{D_N *}\left[\sum_{q=0}^{m_{N-1}} i^*_{D_N}\mathcal{O}_X(-Z_q)\right] = \mathcal{O}_{D'} + i_{D_N *} i^*_{D_N}\left[\mathcal{O}_X(-D')\sum_{q=0}^{m_{N-1}}\mathcal{O}_X\big(-qD_N\big)\right].$$

We choose $\begin{cases} u_N = i_{D_N *} i^*_{D_N}\Big[\mathcal{O}_X(-D')\displaystyle\sum_{q=0}^{m_{N-1}}\mathcal{O}_X\big(-qD_N\big)\Big] \\ u_i = u'_i \qquad \text{for } 1 \leq i \leq N-1. \end{cases}$ Then (a) is true by construction.

Let i be such that $1 \leq i \leq N-1$ and μ' the class defined in Lemma 4.5 for the divisor D'. Then

$$\begin{aligned}
\operatorname{ch}(\overline{u}_i)\operatorname{td}\big(N_{D_i/X}\big) - m_i\, i_{D_i}^*\,\mu &= \operatorname{ch}(\overline{u}'_i)\operatorname{td}\big(N_{D_i/X}\big) - m_i\, i_{D_i}^*\,\mu' + m_i\, i_{D_i}^*\,(\mu'-\mu)\\
&= \sum_{\substack{j=1\\ j\neq i}}^{N-1} i_{D_{ij}\longrightarrow D_i *}\zeta'_{ij} + m_i\ i_{D_i}^*\Bigg[\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots\\
&\qquad\cdots+m_{N-1}\{D_{N-1}\}_D\big)^{k-1-j}\big(m_N\{D_N\}_D\big)^j\Bigg]\\
&\qquad\text{(by induction)}\\
&= \sum_{\substack{j=1\\ j\neq i}}^{N-1} i_{D_{ij}\longrightarrow D_i *}\zeta'_{ij} + m_i\ i_{D_{iN}\longrightarrow D_i *} i_{D_{iN}}^*\Bigg[\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots\\
&\qquad\cdots+m_{N-1}\{D_{N-1}\}_D\big)^{k-1-j}m_N^j\{D_N\}_D^{j-1}\Bigg].
\end{aligned}$$

For the last equality, we have used that

$$i_{D_i}^*\big(\alpha\,\{D_N\}_D\big) = i_{D_i}^*\alpha\,\{D_{iN}\}_D = i_{D_{iN}\longrightarrow D_i *}\big(i_{D_{iN}}^*\alpha\big)$$

where $\{D_{iN}\}_D$ is the cycle class of D_{iN} in D_i. This is a consequence of Proposition 3.4 (i) and Proposition 3.5 (iv).

Let us define

$$\begin{cases}
\zeta_{ij} = \zeta'_{ij} & \text{if } 1\leq i,j\leq N-1,\ i\neq j\\
\zeta_{iN} = m_i\ i_{D_{iN}}^*\Bigg[\displaystyle\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots & \\
\qquad\qquad\cdots+m_{N-1}\{D_{N-1}\}_D\big)^{k-1-j}m_N^j\{D_N\}_D^{j-1}\Bigg] & \text{if } 1\leq i\leq N-1\\
\zeta_{Nj} = -\zeta_{jN} & \text{if } 1\leq j\leq N-1.
\end{cases}$$

Properties (a) and (b) of Lemma 4.5 hold, and property (c) of the same lemma hold for $1 \leq i \leq N-1$. For $i = N$, let us now compute both members of (c). We have

$$\begin{aligned}
\sum_{l=1}^{N-1} i_{D_{Nl}\longrightarrow D_N *}\zeta_{Nl} &= \sum_{l=1}^{N-1} m_l\ i_{D_N}^*\Bigg[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots\\
&\qquad\cdots+m_{N-1}\{D_{N-1}\}\big)^{k-1-j}m_N^j\{D_N\}_D^{j-1}\{D_l\}\Bigg]\\
(*)\qquad &= i_{D_N}^*\Bigg[\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\sum_{j=1}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}_D+\cdots\\
&\qquad\cdots+m_{N-1}\{D_{N-1}\}_D\big)^{k-j}m_N^j\{D_N\}_D^{j-1}\Bigg].
\end{aligned}$$

In the first equality, we have used

$$i_{D_{Nl}\longrightarrow D_N *}\, i_{D_{lN}}^*\alpha = i_{D_N}^*\alpha\,\{D_{lN}\}_D = i_{D_N}^*\,\big(\alpha\,\{D_l\}_D\big),$$

where $\{D_{lN}\}_D$ is the cycle class of D_{lN} in D_N.

Now,

$$\begin{aligned}
&\mathrm{ch}(\overline{u}_N)\,\mathrm{td}\big(N_{D_N/X}\big)^{-1} - m_N\, i^*_{D_N}\,\mu \\
&\quad = i^*_{D_N}\left[e^{-m_1\{D_1\} - \cdots - m_{N-1}\{D_{N-1}\}}\left(\sum_{q=0}^{m_{N-1}} e^{-q\{D_N\}}\right)\frac{1-e^{-\{D_N\}}}{\{D_N\}}\right] \\
&\qquad\qquad - m_N\, i^*_{D_N}\,\mu
\end{aligned}$$

by (C_{n-1}) and Proposition 3.7 (ii) and (iii)

$$\begin{aligned}
&\quad = i^*_{D_N}\left[e^{-m_1\{D_1\} - \cdots - m_{N-1}\{D_{N-1}\}}\ \frac{1-e^{-m_N\{D_N\}}}{\{D_N\}} - m_N\,\mu\right] \\
&\quad = i^*_{D_N}\Bigg[m_N\sum_{r=0}^{\infty}\sum_{q=1}^{\infty}\frac{(-1)^{r+q-1}}{r!\,q!}\big(m_1\{D_1\}+\cdots+m_{N-1}\{D_{N-1}\}\big)^r\big(m_N\{D_N\}\big)^{q-1} \\
&\qquad\qquad - m_N\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k!}\sum_{j=0}^{k-1}\binom{k-1}{j}\big(m_1\{D_1\}+\cdots \\
&\qquad\qquad\qquad \cdots + m_{N-1}\{D_{N-1}\}\big)^{k-1-j}\big(m_N\{D_N\}\big)^j\Bigg].
\end{aligned}$$

In the first term, we put $k = q + r$, $p = q - 1$ and we obtain

$$\begin{aligned}
&m_N\, i^*_{D_N}\Bigg[\sum_{k=1}^{\infty}\sum_{p=0}^{k-1}\frac{(-1)^{k-1}}{k!}\left(\binom{k}{p+1}-\binom{k-1}{p}\right)\big(m_1\{D_1\}+\cdots \\
&\qquad\qquad \cdots + m_{N-1}\{D_{N-1}\}\big)^{k-1-p}\big(m_N\{D_N\}\big)^p\Bigg]. \qquad (**)
\end{aligned}$$

Now $\binom{k}{p+1}-\binom{k-1}{p}$ is equal to $\binom{k-1}{p+1}$ for $p \leq k-2$ and to zero for $p = k-1$. It suffices to put $j = p+1$ in the sum to obtain the equality of $(*)$ and $(**)$. □

Lemma 4.6. *Using the same notations as in Lemma 4.5, let α_i in $H^*_{\mathrm{Del}}(D_i)$, $1 \leq i \leq N$, be such that $i^*_{D_{ij}\to D_i}\alpha_i = i^*_{D_{ij}\to D_j}\alpha_j$. Then there exist u_i in $K_{D_i}(X)$, satisfying $u_1+\cdots+u_N = \mathcal{O}_D$ in $K_{D^{red}}(X)$, such that*

$$\sum_{i=1}^{N} i_{D_i*}\Big(\alpha_i\ \mathrm{ch}(\overline{u}_i)\,\mathrm{td}\big(N_{D_i/X}\big)^{-1}\Big) = \left(\sum_{i=1}^{N} m_i\, i_{D_i*}(\alpha_i)\right)\mu.$$

Proof. We pick $u_1, \ldots, u_N$ given by Lemma 4.5. Then

$$\sum_{i=1}^{N} i_{D_i *}\Big(\alpha_i \operatorname{ch}(\overline{u}_i) \operatorname{td}\big(N_{D_i/X}\big)^{-1}\Big) - \left(\sum_{i=1}^{N} m_i\, i_{D_i *}(\alpha_i)\right)\mu$$
$$= \sum_{i=1}^{N} i_{D_i *}\Big[\alpha_i\Big(\operatorname{ch}(\overline{u}_i) \operatorname{td}\big(N_{D_i/X}\big)^{-1} - m_i\, i^*_{D_i}\, \mu\Big)\Big]$$

(by the projection formula)

$$= \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} i_{D_i *}\Big[\alpha_i\; i_{D_{ij} \longrightarrow D_i *}\zeta_{ij}\Big]$$
$$= \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j\neq i}}^{N} i_{D_{ij} *}\Big(i^*_{D_{ij} \longrightarrow D_i}\alpha_i\; \zeta_{ij}\Big)$$

(by the projection formula).

Grouping the terms (i,j) and (j,i), we get 0, since $\zeta_{ij} = -\zeta_{ji}$. $\square$

We now prove Proposition 4.4.

Proof. (i) We write $x = x_1 + \cdots + x_N$ in $K_{D^{\mathrm{red}}}(X)$, where x_i is an element of $K_{D_i}(X)$. Then

$$\begin{aligned}\operatorname{ch}_D([\mathcal{E}]\,.\,x) &= \sum_{i=1}^{N} \operatorname{ch}_{D_i}([\mathcal{E}]\,.\,x_i) = \sum_{i=1}^{N} \overline{\operatorname{ch}}(\mathcal{E})\,.\operatorname{ch}_{D_i}(x_i) && \text{by Proposition 4.3 (ii)}\\ &= \overline{\operatorname{ch}}(\mathcal{E})\,.\operatorname{ch}_D(x) && \text{by the very definition of } \operatorname{ch}_D(x).\end{aligned}$$

(ii) We choose $u_1, \ldots, u_N$ such that Lemma 4.5 holds. Then

$$\operatorname{ch}\big(\mathcal{O}_{\widetilde{D}}\big) = \sum_{i=1}^{N} \operatorname{ch}(u_i) = \sum_{i=1}^{N} i_{D_i *}\Big(\operatorname{ch}(\overline{u}_i) \operatorname{td}\big(N_{\widetilde{D}_i/X}\big)^{-1}\Big) = \left(\sum_{i=1}^{N} m_i \{\widetilde{D}_i\}_D\right)\mu$$
$$= 1 - e^{-\left(m_1\{\widetilde{D}_1\}_D + \cdots + m_N\{\widetilde{D}_N\}_D\right)} = 1 - \overline{\operatorname{ch}}\big(\mathcal{O}_X(-\widetilde{D})\big).$$

(iii) By dévissage we can suppose that D is a smooth hypersurface of Y. Let $\overline{f}_i$ be defined by the diagram

$$\begin{tikzcd} \widetilde{D_i} \arrow[r] \arrow[d, "\overline{f}_i"'] & X \arrow[d, "f"] \\ D \arrow[r] & Y \end{tikzcd}$$

and let y be an element of $K(D)$. We put $\alpha_i = \overline{f}_i^{\,*} \operatorname{ch}(y)$. By the functoriality property (F_{n-1}) we have $i^*_{\widetilde{D}_{ij} \longrightarrow \widetilde{D}_i}\alpha_i = i^*_{\widetilde{D}_{ij} \longrightarrow \widetilde{D}_i}\alpha_j$. We choose again $u_1, \ldots, u_N$ such that Lemma 4.6 holds.

By Proposition 7.7 of Appendix 7 we can write $f^{\dagger} i_{D*} y = \sum_{i=1}^{N} (\overline{f}_i^{\dagger} y) \cdot_{\widetilde{D}_i} u_i$. Thus

$$\begin{aligned}
\mathrm{ch}_{\widetilde{D}}\big(f^{\dagger} i_{D*} y\big) &= \sum_{i=1}^{N} i_{\widetilde{D}_i *}\Big(\mathrm{ch}\big(\overline{f}_i^{\dagger} y\big)\, \mathrm{ch}(\overline{u}_i)\, \mathrm{td}\big(N_{\widetilde{D}_i/X}\big)^{-1}\Big) && \text{by } (\mathrm{P}_{n-1}) \\
&= \sum_{i=1}^{N} i_{\widetilde{D}_i *}\Big(\alpha_i\, \mathrm{ch}(\overline{u}_i)\, \mathrm{td}\big(N_{\widetilde{D}_i/X}\big)^{-1}\Big) && \text{by } (\mathrm{F}_{n-1}) \\
&= \left(\sum_{i=1}^{N} m_i\ i_{\widetilde{D}_i *}(\alpha_i)\right) \mu && \text{by Lemma 4.6} \\
&= \left[\sum_{i=1}^{N} m_i\ i_{\widetilde{D}_i *}\big(\overline{f}_i^{\,*}\, \mathrm{ch}(y)\big)\right] f^{*} \left(\frac{1 - e^{-\{D\}_D}}{\{D\}_D}\right) \\
&= f^{*}\left[i_{D*}\big(\mathrm{ch}(y)\big) \cdot \frac{1 - e^{-\{D\}_D}}{\{D\}_D}\right] && \text{by Proposition 3.5 (v)} \\
&= f^{*} i_{D*}\Big(\mathrm{ch}(y)\, \mathrm{td}\big(N_{D/Y}\big)^{-1}\Big) && \text{by the projection formula} \\
&= f^{*}\ \mathrm{ch}_D\big(i_{D*} y\big).
\end{aligned}$$

$\square$

(iv) We will first prove it under the assumption that, for all i, either Y and D_i intersect transversally, or $Y = D_i$. By dévissage, we can suppose that D has only one branch and that Y and D intersect transversally, or $Y = D$. We deal with both cases separately.

– If Y and D intersect transversally, then $i_Y^{\dagger}\big[i_{D*}\mathcal{O}_D\big] = \big[i_{Y\cap D \longrightarrow Y *}\mathcal{O}_{Y\cap D}\big]$. Thus, by Proposition 7.5 of Appendix 7,

$$\begin{aligned}
i_Y^{\dagger} x = i_Y^{\dagger}\big(\overline{x} \cdot_D \ [i_{D*}\mathcal{O}_D]\big) &= i^{\dagger}_{Y\cap D \longrightarrow D}\overline{x} \cdot_{Y\cap D}\ \big[i_{Y\cap D \longrightarrow Y *}\mathcal{O}_{Y\cap D}\big] \\
&= i_{Y\cap D \longrightarrow Y *}\big(i^{\dagger}_{Y\cap D \longrightarrow D}\overline{x}\big),
\end{aligned}$$

and we obtain

$$\begin{aligned}
\mathrm{ch}\big(i_Y^{\dagger} x\big) &= i_{Y\cap D \longrightarrow Y *}\Big(\mathrm{ch}\big(i^{\dagger}_{Y\cap D \longrightarrow D}\overline{x}\big)\, \mathrm{td}\big(N_{Y\cap D/Y}\big)^{-1}\Big) && \text{by } (\mathrm{RR}_{n-1}) \\
&= i_{Y\cap D \longrightarrow Y *}\Big(i^{*}_{Y\cap D \longrightarrow D}\, \mathrm{ch}(\overline{x})\ i^{*}_{Y\cap D \longrightarrow D}\, \mathrm{td}\big(N_{D/X}\big)^{-1}\Big) && \text{by } (\mathrm{F}_{n-1}) \\
&= i_Y^{*}\, i_{D*}\Big(\mathrm{ch}(\overline{x})\, \mathrm{td}\big(N_{D/X}\big)^{-1}\Big) && \text{by Proposition 3.5 (iv)} \\
&= i_Y^{*}\, \mathrm{ch}_D(x).
\end{aligned}$$

– If $Y = D$, $i_Y^\dagger\big[i_{D*}\mathcal{O}_D\big] = \big[\mathcal{O}_Y\big] - \big[N^*_{Y/X}\big]$. Thus $i_Y^\dagger\, x = \overline{x} - \overline{x}\,.\,\big[N^*_{Y/X}\big]$ and

$$\begin{aligned}
\mathrm{ch}\big(i_Y^\dagger x\big) &= \mathrm{ch}(\overline{x}) - \mathrm{ch}(\overline{x})\overline{\mathrm{ch}}\big(N^*_{Y/X}\big) && \text{by } (\mathrm{P}_{n-1}) \text{ and } (\mathrm{C}_{n-1})\\
&= \mathrm{ch}(\overline{x})\Big(1 - e^{-c_1\left(N_{Y/X}\right)}\Big)\\
&= \mathrm{ch}(\overline{x})\ \mathrm{td}\big(N_{Y/X}\big)^{-1} c_1\big(N_{Y/X}\big)\\
&= i_Y^* i_{Y*}\Big(\mathrm{ch}(\overline{x})\,\mathrm{td}\big(N_{Y/X}\big)^{-1}\Big) && \text{by Proposition 3.5 (vii)}\\
&= i_Y^*\,\mathrm{ch}_Y(x).
\end{aligned}$$

We examine now the general case. By Hironaka's theorem, we can desingularize $Y \cup D$ by a succession τ of k blowups with smooth centers such that $\tau^{-1}(Y \cup D)$ is a divisor with simple normal crossing. By first blowing up X along Y, we can suppose that $\tau^{-1}(Y) = \breve{D}$ is a subdivisor of $\widetilde{D} = \tau^{-1}(Y \cup D)$. We have the following diagram:

$$\begin{tikzcd}
\breve{D}_j \arrow[r, "i_{\breve{D}_j}"] \arrow[d, "q_j"'] & \widetilde{X} \arrow[d, "\tau"] \\
Y \arrow[r, "i_Y"'] & X
\end{tikzcd}$$

Then

$$\begin{aligned}
q_j^*\,\mathrm{ch}\big(i_Y^\dagger x\big) &= \mathrm{ch}\big(q_j^\dagger i_Y^\dagger x\big) && \text{by } (\mathrm{F}_{n-1})\\
&= \mathrm{ch}\big(i_{\breve{D}_j}^\dagger \tau^\dagger x\big)\\
&= i_{\breve{D}_j}^*\,\mathrm{ch}_{\widetilde{D}}\big(\tau^\dagger x\big) && \text{since } \breve{D}_j \text{ and } \widetilde{D}_i \text{ intersect transversally, or } \breve{D}_j = \widetilde{D}_i\\
&= i_{\breve{D}_j}^*\,\tau^*\,\mathrm{ch}_D(x) && \text{by the first lemma of functoriality 4.4 (iii)}\\
&= q_j^* i_Y^*\,\mathrm{ch}_D(x).
\end{aligned}$$

We can now write q_j as $\delta \circ \mu_j$, where E is the exceptional divisor of the blowup of X along Y, $\delta\colon E \longrightarrow Y$ is the canonical projection and $\mu_j\colon \breve{D}_j \longrightarrow E$ is the restriction of the last $k-1$ blowups to $\breve{D}_j$. Write $\tau = \tau_k \circ \tau_{k-1} \circ \cdots \circ \tau_1$ where τ_i are the blowups. Let us define a sequence of divisors $\big(E_i\big)_{0 \le i \le k}$ by induction: $E_0 = E$, and E_{i+1} is the strict transform of E_i under τ_{i+1}. Since the E_i are smooth divisors, all the maps $\tau_{i+1} : E_{i+1} \longrightarrow E_i$ are isomorphisms. There exists j such that $E_k = \breve{D}_j$. We deduce that $\mu_j = \tau_{|\breve{D}_j} : \breve{D}_j \longrightarrow D$ is an isomorphism. Since δ is the projection of the projective bundle $\mathbb{P}\big(N_{Y/X}\big) \longrightarrow Y$, δ^* is injective. Thus $q_j^* = \mu_j^* \delta^*$ is injective and we get $\mathrm{ch}\big(i_Y^\dagger x\big) = i_Y^*\,\mathrm{ch}_D(x)$. □

Now, we can clear up the problem of the dependence with respect to D of $\mathrm{ch}_D(\mathcal{F})$.

PROPOSITION 4.7. *If D_1 and D_2 are two divisors of X with simple normal crossing such that* $\operatorname{supp}\mathcal{F} \subseteq D_1$ *and* $\operatorname{supp}\mathcal{F} \subseteq D_2$, *then* $\mathrm{ch}_{D_1}(\mathcal{F}) = \mathrm{ch}_{D_2}(\mathcal{F})$.

PROOF. This property is clear if $D_1 \subseteq D_2$. We will reduce the general situation to this case. By Hironaka's theorem, there exists $\tau : \widetilde{X} \longrightarrow X$ such that $\tau^{-1}(D_1 \cup D_2)$ is a divisor with simple normal crossing. Let $\widetilde{D}_1 = \tau^{-1} D_1$ and $\widetilde{D}_2 = \tau^{-1} D_2$. By the first functoriality lemma 4.4 (iii), since $\widetilde{D}_1 \subseteq \widetilde{D}$, we have $\tau^* \operatorname{ch}_{D_1}(\mathcal{F}) = \operatorname{ch}_{\widetilde{D}_1}\big(\tau^\dagger[\mathcal{F}]\big) = \operatorname{ch}_{\widetilde{D}}\big(\tau^\dagger[\mathcal{F}]\big)$. The same property holds for D_2. The map τ is a succession of blowups, thus τ^* is injective and we get $\operatorname{ch}_{D_1}(\mathcal{F}) = \operatorname{ch}_{D_2}(\mathcal{F})$. □

DEFINITION 4.8. If $\operatorname{supp}(\mathcal{F}) \subseteq D$ where D is a normal simple crossing divisor, $\operatorname{ch}(\mathcal{F})$ is defined as $\operatorname{ch}_D(\mathcal{F})$.

By Proposition 4.7, this definition makes sense.

4.1.3. We can now define $\operatorname{ch}(\mathcal{F})$ for an arbitrary torsion sheaf.

Let $\mathcal{F}$ be a torsion sheaf. We say that a succession of blowups with smooth centers $\tau\colon \widetilde{X} \longrightarrow X$ is a desingularization of $\mathcal{F}$ if there exists a divisor with simple normal crossing D in $\widetilde{X}$ such that $\tau^{-1}\big(\operatorname{supp}(\mathcal{F})\big) \subseteq D$. By Hironaka's theorem applied to $\operatorname{supp}(\mathcal{F})$, there always exists such a τ. We say that $\mathcal{F}$ can be desingularized in d steps if there exists a desingularization τ of $\mathcal{F}$ consisting of at most d blowups. In that case, $\operatorname{ch}(\tau^\dagger[\mathcal{F}])$ is defined by Definition 4.8.

PROPOSITION 4.9. *There exists a class* $\operatorname{ch}(\mathcal{F})$ *uniquely determined by* $\mathcal{F}$ *such that*

(i) *If* τ *is a desingularization of* $\mathcal{F}$*, then* $\tau^* \operatorname{ch}(\mathcal{F}) = \operatorname{ch}(\tau^\dagger[\mathcal{F}])$.
(ii) *If* Y *is a smooth submanifold of* X*, then* $\operatorname{ch}\big(i_Y^\dagger[\mathcal{F}]\big) = i_Y^* \operatorname{ch}(\mathcal{F})$.

PROOF. Let d be the number of blowups necessary to desingularize $\mathcal{F}$. Both assertions will be proved at the same time by induction on d.

If $d = 0$, $\operatorname{supp}(\mathcal{F})$ is a subset of a divisor with simple normal crossing D. The properties (i) and (ii) are immediate consequences of the two lemmas of functoriality 4.4 (iii) and (iv).

Suppose now that Proposition 4.9 is proved for torsion sheaves which can be desingularized in $d-1$ steps. Let $\mathcal{F}$ be a torsion sheaf which can be desingularized with at most d blowups. Let $(\widetilde{X}, \tau)$ be such a desingularization. We write τ as $\widetilde{\tau} \circ \tau_1$, where $\widetilde{\tau}$ is the first blowup in τ with E as exceptional divisor, as shown in the following diagram:

$$\begin{tikzcd}
 & \widetilde{X} \arrow[d, "\tau_1"] \\
E \arrow[r, "i_E"] \arrow[d, "q"'] & \widetilde{X}_1 \arrow[d, "\widetilde{\tau}"] \\
Y \arrow[r, "i_Y"'] & X
\end{tikzcd}$$

Then τ_1 consists of at most $d-1$ blowups and is a desingularization of the sheaves $\operatorname{Tor}_j(\mathcal{F}, \widetilde{\tau})$, $0 \leq j \leq n$. By induction, we can consider the following expression in $H^*_{\mathrm{Del}}(\widetilde{X}_1, \mathbb{Q})$:

$$\gamma\big(\widetilde{X}_1, \mathcal{F}\big) = \sum_{j=0}^{n} (-1)^j \operatorname{ch}\Big(\operatorname{Tor}_j(\mathcal{F}, \widetilde{\tau})\Big).$$

We have

$$\begin{aligned}
i_E^*\gamma\big(\widetilde{X}_1, \mathcal{F}\big) &= \sum_{j=0}^{n} (-1)^j \operatorname{ch}\Big(i_E^\dagger\big[\operatorname{Tor}_j(\mathcal{F}, \widetilde{\tau})\big]\Big) && \text{by induction, property (ii)} \\
&= \operatorname{ch}\big(i_E^\dagger \widetilde{\tau}^\dagger[\mathcal{F}]\big) && \text{by } (\mathrm{W}_{n-1}) \\
&= \operatorname{ch}\big(q^\dagger i_Y^\dagger[\mathcal{F}]\big) = q^* \operatorname{ch}\big(i_Y^\dagger[\mathcal{F}]\big) && \text{by } (\mathrm{F}_{n-1}).
\end{aligned}$$

By Proposition 3.5 (vi), there exists a unique class $\mathrm{ch}(\mathcal{F},\tau)$ on X such that $\gamma\big(\widetilde{X}_1,\mathcal{F}\big) = \widetilde{\tau}^*\,\mathrm{ch}\big(\mathcal{F},\tau\big)$. Now

$$\begin{aligned}\tau^*\,\mathrm{ch}(\mathcal{F},\tau) &= \tau_1^*\gamma\big(\widetilde{X}_1,\mathcal{F}\big)\\ &= \sum_{j=0}^{n}(-1)^j\,\mathrm{ch}\Big(\tau_1^{\dagger}\big[\mathrm{Tor}_j(\mathcal{F},\widetilde{\tau})\big]\Big) && \text{by induction, property (i)}\\ &= \mathrm{ch}\big(\tau_1^{\dagger}\widetilde{\tau}^{\dagger}[\mathcal{F}]\big) = \mathrm{ch}\big(\tau^{\dagger}[\mathcal{F}]\big).\end{aligned}$$

Suppose that we have two resolutions. We dominate them by a third one, according to the diagram:

$$\begin{tikzcd} & W \arrow[dl, "\mu"'] \arrow[dr, "\breve{\mu}"] \arrow[dd, "\delta"] & \\ \widetilde{X}_1 \arrow[dr, "\tau"'] & & \widetilde{X}_2 \arrow[dl, "\breve{\tau}"] \\ & X & \end{tikzcd}$$

Then

$$\begin{aligned}\delta^*\,\mathrm{ch}(\mathcal{F},\tau) &= \mu^*\,\mathrm{ch}\big(\tau^{\dagger}[\mathcal{F}]\big)\\ &= \mathrm{ch}\big(\mu^{\dagger}\tau^{\dagger}[\mathcal{F}]\big) && \text{by the first lemma of functoriality 4.4 (iii)}\\ &= \mathrm{ch}\big(\delta^{\dagger}[\mathcal{F}]\big) = \delta^*\,\mathrm{ch}(\mathcal{F},\breve{\tau}) && \text{by symmetry.}\end{aligned}$$

The map δ^* being injective, $\mathrm{ch}(\mathcal{F},\tau) = \mathrm{ch}(\mathcal{F},\breve{\tau})$, and we can therefore define $\mathrm{ch}(\mathcal{F})$ by $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{F},\tau)$ for any desingularization τ of $\mathcal{F}$ with at most d blowups.

We have shown that (i) is true when τ consists of at most d blowups. In the general case, let $\tau\colon \widetilde{X}_1 \longrightarrow X$ be an arbitrary desingularization of $\mathcal{F}$ and $\breve{\tau}$ be a desingularization of $\mathcal{F}$ with at most d blowups. We can find W, μ and $\breve{\mu}$ as before. Then

$$\begin{aligned}\mu^*\,\mathrm{ch}\big(\tau^{\dagger}[\mathcal{F}]\big) &= \mathrm{ch}\big(\delta^{\dagger}[\mathcal{F}]\big) && \text{by the first functoriality lemma 4.4 (iii)}\\ &= \breve{\mu}^*\,\mathrm{ch}\big(\breve{\tau}^{\dagger}[\mathcal{F}]\big) && \text{by the first functoriality lemma 4.4 (iii)}\\ &= \breve{\mu}^*\breve{\tau}^*\,\mathrm{ch}(\mathcal{F}) && \text{since } \breve{\tau} \text{ consists of at most } d \text{ blowups}\\ &= \mu^*\tau^*\,\mathrm{ch}(\mathcal{F}).\end{aligned}$$

It remains to show (ii). For this, we desingularize $\mathrm{supp}(\mathcal{F})\cup Y$ exactly as in the proof of the second lemma of functoriality 4.4 (iv). We have a diagram

$$\begin{tikzcd} \breve{D}_i \arrow[r, "i_{\breve{D}_i}"'] \arrow[d, "q_i"'] & \widetilde{X} \arrow[d, "\tau"] \\ Y \arrow[r, "i_Y"'] & X \end{tikzcd}$$

where q_i^* is injective for at least one i. Then

$$\begin{aligned}q_i^*\big(i_Y^*\,\mathrm{ch}(\mathcal{F})\big) &= i_{\breve{D}_i}^*\tau^*\,\mathrm{ch}(\mathcal{F}) = i_{\breve{D}_i}^*\,\mathrm{ch}(\tau^{\dagger}[\mathcal{F}]) && \text{by (i)}\\ &= \mathrm{ch}\big(i_{\breve{D}_i}^{\dagger}\tau^{\dagger}[\mathcal{F}]\big) && \text{by the second lemma of functoriality 4.4 (iv)}\\ &= \mathrm{ch}\big(q_i^{\dagger}\,i_Y^{\dagger}[\mathcal{F}]\big) = q_i^*\,\mathrm{ch}\big(i_Y^{\dagger}[\mathcal{F}]\big) && \text{by } (\mathrm{F}_{n-1}).\end{aligned}$$

Thus $i_Y^* \operatorname{ch}(\mathcal{F}) = \operatorname{ch}\big(i_Y^\dagger\,[\mathcal{F}]\big)$, which proves (ii). $\square$

We have now completed the existence part of Theorem 4.1 for torsion sheaves.

We turn to the proof of Proposition 4.2. So doing, we establish almost all the properties listed in the induction hypotheses for torsion sheaves.

PROOF. (i) Let $(\widetilde{X},\tau)$ be a desingularization of $\operatorname{supp}(\mathcal{F})\cup\operatorname{supp}(\mathcal{H})$ and D be the associated simple normal crossing divisor. Then $\tau^\dagger\mathcal{F}$, $\tau^\dagger\mathcal{G}$ and $\tau^\dagger\mathcal{H}$ belong to $K_D(\widetilde{X})$ and $\tau^\dagger\mathcal{F}+\tau^\dagger\mathcal{H}=\tau^\dagger\mathcal{G}$ in $K_D(X)$. Thus, by Proposition 4.9 (i),

$$\tau^*\big[\operatorname{ch}(\mathcal{F})+\operatorname{ch}(\mathcal{H})\big]=\operatorname{ch}\big(\tau^\dagger[\mathcal{F}]\big)+\operatorname{ch}\big(\tau^\dagger[\mathcal{H}]\big)=\operatorname{ch}\big(\tau^\dagger[\mathcal{G}]\big)=\tau^*\operatorname{ch}(\mathcal{G}).$$

The map τ^* being injective, we get the Whitney formula for torsion sheaves.

(ii) The method is the same: let $x=[\mathcal{G}]$ and let τ be a desingularization of $\mathcal{G}$. Then, by Proposition 4.9 (i) and Proposition 4.4 (i),

$$\tau^*\operatorname{ch}([\mathcal{E}]\,.\,[\mathcal{G}])=\operatorname{ch}\big(\tau^\dagger[\mathcal{E}]\,.\,\tau^\dagger[\mathcal{G}]\big)=\overline{\operatorname{ch}}\big(\tau^\dagger[\mathcal{E}]\big)\,.\,\operatorname{ch}\big(\tau^\dagger[\mathcal{G}]\big)=\tau^*\big(\overline{\operatorname{ch}}(\mathcal{E})\,.\,\operatorname{ch}(\mathcal{G})\big).$$

(iii) This property is known when f if the immersion of a smooth submanifold and when f is a bimeromorphic morphism by Proposition 4.9. Let us consider now the general case. By Grauert's direct image theorem, $f(X)$ is an irreducible analytic subset of Y. We desingularize $f(X)$ as an abstract complex space. We get a connected smooth manifold W and a bimeromorphic morphism $\tau\colon W\longrightarrow f(X)$ obtained as a succession of blowups with smooth centers in $f(X)$. We perform a similar sequence of blowups, starting from $Y_1=Y$ and blowing up at each step in Y_i the smooth center blown up at the i-th step of the desingularization of $f(X)$. Let $\pi_Y\colon\widetilde{Y}\longrightarrow Y$ be this morphism. The strict transform of $f(X)$ is W. The map $\tau:\tau^{-1}\big(f(X)_{\mathrm{reg}}\big)\xrightarrow{\ \sim\ }f(X)_{\mathrm{reg}}$ is an isomorphism. So we get a morphism $f(X)_{\mathrm{reg}}\longrightarrow W$ which is in fact a meromorphic map from $f(X)$ to W, and finally, after composition on the left by f, from X to W. We desingularize this morphism:

$$\begin{tikzcd}
\widetilde{X} \arrow[d, "\pi_X"'] \arrow[dr, "\widetilde{f}"] & \\
X \arrow[r, dotted] & W
\end{tikzcd}$$

and we get the following global diagram, where π_X is a bimeromorphic map:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_X"'] & W \arrow[r, "i_W"] \arrow[d, "\tau"] & \widetilde{Y} \arrow[d, "\pi_Y"] \\
X \arrow[r, "f"'] & f(X) \arrow[r] & Y
\end{tikzcd}$$

Now $f\circ\pi_X=\pi_Y\circ\big(i_W\circ\widetilde{f}\,\big)$, and we know the functoriality formula for π_X, π_Y and i_W by Proposition 4.9. Since π_X^* is injective, it is enough to show the functoriality formula for $\widetilde{f}$. So we will assume that f is onto.

Let $(\tau, \widetilde{Y})$ be a desingularization of $\mathcal{F}$. We have the diagram

$$\begin{tikzcd} X \times_Y \widetilde{Y} \arrow[r] \arrow[d] & \widetilde{Y} \arrow[d, "\tau"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

where $\widetilde{\tau}^{-1}(\operatorname{supp}\mathcal{F}) = D \subseteq \widetilde{Y}$ is a divisor with simple normal crossing and $X \times_Y \widetilde{Y} \longrightarrow X$ is a bimeromorphic morphism. We have a meromorphic map $X \dashrightarrow X \times_Y \widetilde{Y}$, and we desingularize it by a morphism $T \longrightarrow X \times_Y \widetilde{Y}$. Therefore, we obtain the following commutative diagram, where $\pi : T \longrightarrow X$ is a bimeromorphic map:

$$\begin{tikzcd} T \arrow[r, "\widetilde{f}"] \arrow[d, "\pi"'] & \widetilde{Y} \arrow[d, "\tau"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Therefore we can assume that supp($\mathcal{F}$) is included in a divisor with simple normal crossing D. We desingularize $f^{-1}(D)$ so that we are led to the case supp($\mathcal{F}$) $\subseteq D$, where D and $f^{-1}(D)$ are divisors with simple normal crossing in Y and X respectively. In this case, we can use the first lemma of functoriality 4.4 (iii). The proof of Proposition 4.2 is finished, since (iv) holds by construction. □

4.2. The dévissage theorem for sheaves of positive rank. In this section, we consider the case of sheaves of arbitrary rank. Let X be a complex compact manifold and $\mathcal{F}$ an analytic coherent sheaf on X. We have seen in section 4.1 how to define ch($\mathcal{F}$) when $\mathcal{F}$ is a torsion sheaf.

Suppose that $\mathcal{F}$ has strictly positive generic rank. When $\mathcal{F}$ admits a global locally free resolution, we could try to define ch($\mathcal{F}$) the usual way. As explained in the introduction, this condition on $\mathcal{F}$ is not necessarily fulfilled. Even if such a resolution exists, the definition of ch($\mathcal{F}$) depends a priori on this resolution. A good substitute for a locally free resolution is a locally free quotient with maximal rank, since the kernel is then a torsion sheaf. Let $\mathcal{F}_{\mathrm{tor}} \subseteq \mathcal{F}$ be the maximal torsion subsheaf of $\mathcal{F}$. Then $\mathcal{F}$ admits a locally free quotient $\mathcal{E}$ of maximal rank if and only if $\mathcal{F} \big/ \mathcal{F}_{\mathrm{tor}}$ is locally free. In this case, $\mathcal{E} = \mathcal{F} \big/ \mathcal{F}_{\mathrm{tor}}$.

Unfortunately, the existence of such a quotient is not assured (for instance, take a torsion-free sheaf which is not locally free), but it exists up to a bimeromorphic morphism thanks to the following result:

THEOREM 4.10. [**Ro**] *Let X be a complex compact manifold and $\mathcal{F}$ a coherent analytic sheaf on X. There exists a bimeromorphic morphism $\sigma : \widetilde{X} \longrightarrow X$, which is a finite composition of blowups with smooth centers, such that $\sigma^*\mathcal{F}$ admits a locally free quotient of maximal rank on $\widetilde{X}$. Such quotients are unique, up to a unique isomorphism.*

PROOF. Let r be the rank of $\mathcal{F}$. We define a universal set $\widetilde{X}$ by $\widetilde{X} = \coprod_{x \in X} \mathrm{Gr}^*\big(r, \mathcal{F}_{|x}\big)$ where $\mathrm{Gr}^*\big(r, \mathcal{F}_{|x}\big)$ is the dual grassmannian of quotients of $\mathcal{F}_{|x}$ with rank r. The set $\widetilde{X}$ is the disjoint union of all the quotients of rank r of all fibers of $\mathcal{F}$. The canonical map $\sigma : \widetilde{X} \longrightarrow X$ is a bijection on $\sigma^{-1}\big(\mathcal{F}_{\mathrm{reg}}\big)$. We will now endow $\widetilde{X}$ with the structure of a reduced complex space.

Let us first argue locally. Let $\mathcal{O}^p_{|U} \xrightarrow{M} \mathcal{O}^q_{|U} \longrightarrow \mathcal{F}_{|U} \longrightarrow 0$ be a presentation of $\mathcal{F}$ on an open set U. Here, M is an element of $\mathfrak{M}_{q,p}\big(\mathcal{O}_U\big)$. Then, for every x in U, we get the exact sequence

$$\mathbb{C}^p \xrightarrow{M(x)} \mathbb{C}^q \xrightarrow{\pi_x} \mathcal{F}_{|x} \longrightarrow 0.$$

We have an inclusion

$$\mathrm{Gr}^*\big(r, \mathcal{F}_{|x}\big) \hookrightarrow \mathrm{Gr}^*\big(r, \mathbb{C}^q\big) \xrightarrow{\sim} \mathrm{Gr}(q-r, q)$$

given by

$$(\overline{q}, Q) \longmapsto (\overline{q} \circ \pi_x, Q) \longmapsto \ker(\overline{q} \circ \pi_x),$$

where $\overline{q}$ and Q appear in the following diagram:

$$\begin{tikzcd} \mathbb{C}^p \arrow[r, "M(x)"] & \mathbb{C}^q \arrow[r, "\pi_x"] \arrow[dr, dotted, "\widetilde{q}"'] & \mathcal{F}_{|x} \arrow[r] \arrow[d, "\overline{q}"] & 0 \\ & & Q \arrow[d] & \\ & & 0 & \end{tikzcd}$$

Therefore, we have a fibered inclusion $\sigma^{-1}(U) \hookrightarrow U \times \mathrm{Gr}(q-r, q)$. We know that

$$\sigma^{-1}(U) = \big\{(x, E) \in U \times \mathrm{Gr}(q-r, q) \text{ such that } \operatorname{Im} M(x) \subseteq E\big\}\cdot$$

Let $(e_1, \dots, e_q)$ be the canonical basis of $\mathbb{C}^q$, and $e_1^*, \dots, e_q^*$ its dual basis. We can suppose that $e_1^*, \dots, e_{q-r}^*$ are linearly independent on E. We parametrize $\mathrm{Gr}(q-r, q)$ in the neighborhood of x by a matrix $A = \big(a_{i,j}\big) \in \mathfrak{M}_{r,q-r}(\mathbb{C})$. The associated vector space will be spanned by the columns of the matrix $\begin{pmatrix} \mathrm{id}_{q-r} \\ A \end{pmatrix}$. Writing $M = \big(M_j^i\big)_{\substack{1 \le j \le q \\ 1 \le i \le p}}$, $\mathrm{Im}(x)$ is a subspace of E if and only if for all i, $1 \le i \le p$,

$$\Big(M_1^i(x) e_1 + \cdots + M_q^i(x) e_q\Big) \wedge \bigwedge_{l=1}^{q-r} \big(e_l + a_{1,l} e_{q-r+1} + \cdots + a_{r,l} e_q\big) = 0$$

in $\bigwedge^{q-r+1} \mathbb{C}^q$. This is clearly an analytic condition in the variables x and a_{ij}, thus $\sigma^{-1}(U)$ is an analytic subset of $U \times \mathrm{Gr}(q-r, q)$. We endow $\sigma^{-1}(U)$ with the associated *reduced* structure.

We must check carefully that the structure defined above does not depend on the chosen presentation.

Let us consider two resolutions of $\mathcal{F}$ on U

$$\mathcal{O}^p_{|U} \xrightarrow{M} \mathcal{O}^q_{|U} \xrightarrow{\pi} \mathcal{F}_{|U} \longrightarrow 0$$

$$\mathcal{O}^{p'}_{|U} \xrightarrow{M'} \mathcal{O}^{q'}_{|U} \xrightarrow{\pi'} \mathcal{F}_{|U} \longrightarrow 0$$

and a (q',q) matrix $\alpha\colon \mathcal{O}^q_{|U} \longrightarrow \mathcal{O}^{q'}_{|U}$ such that the diagram

$$\begin{tikzcd}
\mathcal{O}^p_{|U} \arrow[r, "M"] & \mathcal{O}^q_{|U} \arrow[r, "\pi"] \arrow[d, "\alpha"] & \mathcal{F}_{|U} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\mathcal{O}^{p'}_{|U} \arrow[r, "M'"] & \mathcal{O}^{q'}_{|U} \arrow[r, "\pi'"] & \mathcal{F}_{|U} \arrow[r] & 0
\end{tikzcd}$$

commutes. The morphism

$$\begin{tikzcd}
\sigma^{-1}(U) \arrow[r, "\mathrm{id}"] \arrow[d, hook] & \sigma^{-1}(U) \arrow[d, hook] \\
U \times \mathrm{Gr}(q-r,q) \arrow[r] & U \times \mathrm{Gr}(q'-r,q')
\end{tikzcd}$$

is given by $(x,E) \longmapsto \big(x, \pi_x'^{-1}\pi_x(E)\big)$ according to the following diagram:

$$\begin{tikzcd}
\mathbb{C}^p \arrow[r, "M(x)"] & \mathbb{C}^q \arrow[r, "\pi_x"] \arrow[d, "\alpha(x)"] & \mathcal{F}_{|x} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
\mathbb{C}^{p'} \arrow[r, "M'(x)"'] & \mathbb{C}^{q'} \arrow[r, "\pi'_x"'] & \mathcal{F}_{|x} \arrow[r] & 0
\end{tikzcd}$$

Since $\pi'_x\big(\alpha(x)(E)\big) = \pi_x(E)$, we have $\pi_x'^{-1}\pi_x(E) = \pi_x'^{-1}\pi'_x\big(\alpha(x)(E)\big)$. We can write this

$$\pi_x'^{-1}\pi_x(E) = \alpha(x)(E) + \ker \pi'_x = \alpha(x)(E) + \operatorname{Im} M'(x).$$

If (x_0, E_0) is an element of $\sigma^{-1}(U) \times \mathrm{Gr}(q-r,q)$, then $\alpha(x_0)(E_0) + \operatorname{Im} M'(x_0)$ belongs to $\mathrm{Gr}(q'-r,q')$. We can suppose as above that $e_1^*, \ldots, e_{q-r}^*$ are linearly independent on E_0. Therefore, E_0 is spanned by $q-r$ vectors $A_1, \ldots, A_{q-r}$ where

$$A_i = \big(0, \ldots, 1, \ldots, a_{1,i}, \ldots, a_{q-r,i}\big)^{\mathrm{T}},$$

the first "1" having index i. Then $\alpha(x_0)(E_0) + \operatorname{Im} M'(x_0)$ is spanned by the vectors $\big(\alpha(x_0)(A_i) + M'^j(x_0)\big)_{\substack{1 \le i \le q-r \\ 1 \le j \le p'}}$, where the M'^j are the columns of M'. We can find $q'-r$ independent vectors in this family, and this property holds also in a neighborhood of (x_0, E_0). We denote these vectors by $\big(\alpha(x)(A_{i_k}) + M'^{j_k}(x)\big)_{1 \le k \le q'-r}$. Let us define

$$f\big(A_1, \ldots, A_{q-r}, x\big) = \operatorname{span}\big(\alpha(x)(A_{i_k}) + M'^{j_k}(x)\big)_{1 \le k \le q'-r}.$$

This is a holomorphic map from a neighborhood of (x_0, E_0) to $U \times \mathrm{Gr}(q'-r, q')$. On the same pattern, we can define another map g on a neighborhood of $f\big(x_0, E_0\big)$ with values in $U \times \mathrm{Gr}(q-r,q)$. The couple (f,g) defines an isomorphism of complex spaces. This proves that $\widetilde{X}$ is endowed with the structure of an intrinsic reduced complex analytic space (not generally smooth).

We define a subsheaf $\mathcal{N}$ of $\sigma^*\mathcal{F}$ by

$$\mathcal{N}(V) = \big\{ s \in \sigma^*\mathcal{F}(V) \text{ such that } \forall \big(x, \{\overline{q}, Q\}\big) \in V,\ s_x \in \ker \overline{q} \big\}.$$

Remark that $\mathcal{N}$ is supported in the singular locus of $\sigma^*\mathcal{F}$.

Lemma 4.11. *$\mathcal{N}$ satisfies the following properties:*

(i) *$\mathcal{N}$ is a coherent subsheaf of $\sigma^*\mathcal{F}$.*
(ii) *$\sigma^*\mathcal{F}/_{\mathcal{N}}$ is locally free and* $\operatorname{rank}\big(\sigma^*\mathcal{F}/_{\mathcal{N}}\big) = \operatorname{rank}(\mathcal{F})$.

Proof. We take a local presentation $\mathcal{O}^p_{|U} \xrightarrow{M} \mathcal{O}^q_{|U} \xrightarrow{\pi} \mathcal{F}_{|U} \longrightarrow 0$ of $\mathcal{F}$. Then $\sigma^*\mathcal{F}$ has the presentation

$$\mathcal{O}^p_{|\sigma^{-1}(U)} \xrightarrow{M\circ\sigma} \mathcal{O}^q_{|\sigma^{-1}(U)} \longrightarrow \sigma^*\mathcal{F}_{|\sigma^{-1}(U)} \longrightarrow 0.$$

Let $(x,E) \longmapsto \big(f_1(x,E),\dots,f_q(x,E)\big)$ be a section of $\sigma^*\mathcal{F}$ on $V \subseteq \sigma^{-1}(U)$. Then s is a section of $\mathcal{N}$ if and only if for every (x,E) in V, $\big(f_1(x,E),\dots,f_q(x,E)\big)$ is an element of E. Let $U_{q-r,q}$ be the universal bundle on $\mathrm{Gr}(q-r,q)$. Then $U_{q-r,q\,|V}$ is a subbundle of $\mathcal{O}^q_{|V}$ and $\mathcal{N}_{|V}$ is the image of $U_{q-r,q\,|V}$ by the morphism $U_{q-r,q\,|V} \hookrightarrow \mathcal{O}^q_{|\sigma^{-1}(U)} \xrightarrow{\pi} \sigma^*\mathcal{F}_{|\sigma^{-1}(U)}$. So $\mathcal{N}$ is coherent.

(ii) Let us define $\mathcal{E}$ by $\mathcal{E} = \sigma^*\mathcal{F}/_{\mathcal{N}}$. For all (x,E) in V, we have an exact sequence

$$\mathcal{N}_{|(x,E)} \longrightarrow \mathcal{F}_{|x} \longrightarrow \mathcal{E}_{|(x,E)} \longrightarrow 0$$

and a commutative diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & \mathbb{C}^q \arrow[d, "\pi_x"] \\ \mathcal{N}_{|(x,E)} \arrow[r] & \mathcal{F}_{|x} \end{tikzcd}$$

The first vertical arrow is the morphism $U_{q-r,q\,|U} \longrightarrow \mathcal{N}$ restricted at (x,E), so it is onto. Thus $\pi_x(E)$ is the image of the morphism $\mathcal{N}_{|(x,E)} \longrightarrow \mathcal{F}_{|x}$. Since we have an exact sequence

$$0 \longrightarrow \pi_x(E) \longrightarrow \mathcal{F}_{|x} \xrightarrow{\overline{q}} Q \longrightarrow 0$$

where $(x,E) = (x,Q)$, then $\dim \pi_x(E) = \dim \mathcal{F}_{|x} - r$. Since $\dim \mathcal{F}_{|x} = \dim \pi_x(E) + \dim \mathcal{E}_{|(x,E)}$, we get $\dim \mathcal{E}_{|(x,E)} = r$. We can see that $\mathcal{N}$ is a torsion sheaf, for $\mathcal{E}$ is locally free of rank r. □

We can now finish the proof of Theorem 4.10. Using Hironaka's theorem, we desingularize the complex space $\widetilde{X}$. We get a succession of blowups with smooth centers $\widetilde{\sigma}\colon \widetilde{X}' \longrightarrow \widetilde{X}$ where $\widetilde{X}'$ is smooth. By the Hironaka-Chow lemma (see [**An-Ga**, Th. 7.8]), we can suppose that $\tau = \sigma \circ \widetilde{\sigma}$ is a succession of blowups with smooth centers. Since $\mathcal{E}$ is locally free, the following sequence is exact:

$$0 \longrightarrow \widetilde{\sigma}^*\mathcal{N} \longrightarrow \tau^*\mathcal{F} \longrightarrow \widetilde{\sigma}^*\mathcal{E} \longrightarrow 0\,.$$

Therefore $\tau^*\mathcal{F}$ admits a locally free quotient of maximal rank. The unicity is clear. This finishes the proof. □

4.3. Construction of the classes in the general case. Let X be a complex compact manifold of dimension n.

4.3.1. Let $\mathcal{F}$ be a coherent sheaf on X which has a locally free quotient of maximal rank. We have an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{T}$ is a torsion sheaf and $\mathcal{E}$ is locally free. Then we define $\mathrm{ch}(\mathcal{F})$ by $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}(\mathcal{T}) + \overline{\mathrm{ch}}(\mathcal{E})$, where $\mathrm{ch}(\mathcal{T})$ has been constructed in part 4.1. Remark that $\mathrm{ch}(\mathcal{F})$ depends only on $\mathcal{F}$, the exact sequence $0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$ being unique up to (a unique) isomorphism.

We state now the Whitney formulae which apply to the Chern characters we have defined above.

PROPOSITION 4.12. *Let* $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ *be an exact sequence of coherent analytic sheaves on* X. *Then* $\mathrm{ch}(\mathcal{F})$, $\mathrm{ch}(\mathcal{G})$ *and* $\mathrm{ch}(\mathcal{H})$ *are well defined and verify* $\mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$ *under any of the following hypotheses:*

(i) $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ *are locally free sheaves on* X.
(ii) $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ *are torsion sheaves.*
(iii) $\mathcal{G}$ *admits a locally free quotient of maximal rank and* $\mathcal{F}$ *is a torsion sheaf.*

PROOF. (i) If $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ are locally free sheaves on X, then $\mathrm{ch}(\mathcal{F}) = \overline{\mathrm{ch}}(\mathcal{F})$, $\mathrm{ch}(\mathcal{G}) = \overline{\mathrm{ch}}(\mathcal{G})$, $\mathrm{ch}(\mathcal{H}) = \overline{\mathrm{ch}}(\mathcal{H})$ and we use Proposition 3.7 (i).

(ii) This is Proposition 4.2 (i).

(iii) Let $\mathcal{E}$ be the locally free quotient of maximal rank of $\mathcal{G}$. We have an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{T}$ is a torsion sheaf. Since $\mathcal{F}$ is a torsion sheaf, the morphism $\mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}$ is identically zero. Let us define $\mathcal{T}'$ by the exact sequence

$$0 \longrightarrow \mathcal{T}' \longrightarrow \mathcal{H} \longrightarrow \mathcal{E} \longrightarrow 0.$$

Then $\mathcal{T}'$ is a torsion sheaf and we have the exact sequence of torsion sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}' \longrightarrow 0.$$

Thus $\mathcal{H}$ admits a locally free quotient of maximal rank, so that $\mathrm{ch}(\mathcal{H})$ is defined and

$$\begin{aligned} \mathrm{ch}(\mathcal{H}) &= \overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}(\mathcal{T}') = \overline{\mathrm{ch}}(\mathcal{E}) + \mathrm{ch}(\mathcal{T}) - \mathrm{ch}(\mathcal{F}) \qquad \text{by (ii)} \\ &= \mathrm{ch}(\mathcal{G}) - \mathrm{ch}(\mathcal{F}). \end{aligned}$$

□

Let us now look at the functoriality properties with respect to pullbacks.

PROPOSITION 4.13. *Let* $f\colon X \longrightarrow Y$ *be a holomorphic map. We assume that*

– $\dim Y = n$ *and* $\dim X \leq n$,
– *if* $\dim X = n$, f *is surjective.*

Then for every coherent sheaf on Y *which admits a locally free quotient of maximal rank, the following properties hold:*

(i) *The Chern characters* $\mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{F}, f)\big)$ *are well defined.*
(ii) $f^*\,\mathrm{ch}(\mathcal{F}) = \sum_{i\geq 0} (-1)^i\,\mathrm{ch}\big(\mathrm{Tor}_i(\mathcal{F}, f)\big)$.

Proof. (i) If $\dim X < n$, the classes $\operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F}, f)\big)$ are defined by the induction property (E_{n-1}). If $\dim X = n$ and f is surjective, then f is generically finite. Thus all the sheaves $\operatorname{Tor}_i(\mathcal{F}, f)$, $i \geq 1$, are torsion sheaves on X, so their Chern classes are defined by Proposition 4.2. The sheaf $f^*\mathcal{F}$ admits on X a locally free quotient of maximal rank so that $\operatorname{ch}(f^*\mathcal{F})$ is well defined.

(ii) We have an exact sequence

$$0 \longrightarrow \mathcal{T} \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{T}$ is a torsion sheaf and $\mathcal{E}$ is a locally free sheaf. Remark that, for $i \geq 1$, $\operatorname{Tor}_i(\mathcal{F}, f) \simeq \operatorname{Tor}_i(\mathcal{T}, f)$. Thus, by Proposition 3.7 (ii) and Proposition 4.2 (iii),

$$\begin{aligned}\sum_{i\geq 0}(-1)^i \operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F}, f)\big) &= \overline{\operatorname{ch}}\big(f^*\mathcal{E}\big) + \operatorname{ch}\big(f^*\mathcal{T}\big) + \sum_{i\geq 1}(-1)^i \operatorname{ch}\Big(\operatorname{Tor}_i(\mathcal{T}, f)\Big)\\ &= f^*\overline{\operatorname{ch}}(\mathcal{E}) + \operatorname{ch}\big(f^{\dagger}[\mathcal{T}]\big) = f^*\big(\overline{\operatorname{ch}}(\mathcal{E}) + \operatorname{ch}(\mathcal{T})\big) = f^*\operatorname{ch}(\mathcal{F}).\end{aligned}$$

$\square$

4.3.2. We consider now an arbitrary coherent sheaf $\mathcal{F}$ on X. By Theorem 4.10, there exists $\sigma\colon \widetilde{X} \longrightarrow X$ obtained as a finite composition of blowups with smooth centers such that $\sigma^*\mathcal{F}$ admits a locally free quotient of maximal rank. This is the key property for the definition of $\operatorname{ch}(\mathcal{F})$ in full generality.

Theorem 4.14. *There exists a class $\operatorname{ch}(\mathcal{F})$ on X uniquely determined by $\mathcal{F}$ such that:*

(i) *If $\sigma\colon \widetilde{X} \longrightarrow X$ is a succession of blowups with smooth centers such that $\sigma^*\mathcal{F}$ admits a locally free quotient of maximal rank, then*

$$\sigma^*\operatorname{ch}(\mathcal{F}) = \sum_{i\geq 0}(-1)^i \operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F}, \sigma)\big).$$

(ii) *If Y is a smooth submanifold of X, $\operatorname{ch}\big(i_Y^{\dagger}[\mathcal{F}]\big) = i_Y^*\operatorname{ch}(\mathcal{F})$.*

Remark 4.15. By Proposition 4.13 (i), all the terms in (i) are defined.

Proof. The proof of (i) will use Lemmas 4.16, 4.17, and 4.18. We will prove the result by induction on the number d of blowups in σ. If $d = 0$, $\mathcal{F}$ admits a locally free quotient of maximal rank and we can use Proposition 4.13.

Suppose now that (i) and (ii) hold at step $d-1$. As usual, we look at the first blowup in σ

$$\begin{tikzcd} & \widetilde{X} \arrow[d, "\sigma_1"] \arrow[dd, bend left, "\sigma"] \\ E \arrow[r, "i_E"] \arrow[d, "q"'] & \widetilde{X}_1 \arrow[d, "\widetilde{\sigma}"] \\ Y \arrow[r, "i_Y"'] & X. \end{tikzcd}$$

The sheaves $\operatorname{Tor}_j(\mathcal{F}, \sigma)$ are torsion sheaves for $j \geq 1$ and $\sigma_1^*\operatorname{Tor}_0(\mathcal{F}, \widetilde{\sigma}) = \sigma^*\mathcal{F}$ admits a locally free quotient of maximal rank. Since σ_1 consists of $d-1$ blowups, we can define by induction a class $\gamma(\mathcal{F})$ in $H^*_{\mathrm{Del}}(\widetilde{X}_1, \mathbb{Q})$ as follows:

$$\gamma(\mathcal{F}) = \sum_{j\geq 0}(-1)^j \operatorname{ch}\Big(\operatorname{Tor}_j(\mathcal{F}, \widetilde{\sigma})\Big).$$

LEMMA 4.16. $\sigma_1^*\gamma(\mathcal{F}) = \sum_{i\geq 0}(-1)^i \operatorname{ch}\bigl(\operatorname{Tor}_i(\mathcal{F},\sigma)\bigr)$.

PROOF. We have by induction

$$\sigma_1^*\gamma(\mathcal{F}) = \sum_{p,q\geq 0}(-1)^{p+q}\operatorname{ch}\Bigl[\operatorname{Tor}_p(\operatorname{Tor}_q(\mathcal{F},\widetilde{\sigma}),\sigma_1)\Bigr] = \sum_{p,q\geq 0}(-1)^{p+q}\operatorname{ch}(E_2^{p,q})$$

where the Tor spectral sequence satisfies

$$\begin{aligned} E_2^{p,q} &= \operatorname{Tor}_p(\operatorname{Tor}_q(\mathcal{F},\widetilde{\sigma}),\sigma_1)\\ E_\infty^{p,q} &= \operatorname{Gr}^p\ \operatorname{Tor}_{p+q}(\mathcal{F},\sigma). \end{aligned}$$

All the $E_r^{p,q}$, $2\leq r\leq\infty$, are torsion sheaves except perhaps $E_r^{0,0}$. Remark that no arrow $d_r^{p,q}$ starts or arrives at $E_r^{0,0}$. Thus we have

$$\sum_{\substack{p,q\\ p+q\geq 1}}(-1)^{p+q}[E_2^{p,q}] = \sum_{\substack{p,q\\ p+q\geq 1}}(-1)^{p+q}[E_\infty^{p,q}] = \sum_{i\geq 1}(-1)^i\bigl[\operatorname{Tor}_i(\mathcal{F},\sigma)\bigr]$$

in $K_{\mathrm{tors}}(X)$. Using Proposition 4.12 (ii), we get

$$\begin{aligned} \sigma_1^*\gamma(\mathcal{F}) &= \operatorname{ch}\bigl(E_2^{0,0}\bigr) + \operatorname{ch}\Bigl(\sum_{i\geq 1}(-1)^i\operatorname{Tor}_i(\mathcal{F},\sigma)\Bigr)\\ &= \sum_{i\geq 0}(-1)^i\operatorname{ch}\Bigl(\operatorname{Tor}_i(\mathcal{F},\sigma)\Bigr). \end{aligned}$$

□

LEMMA 4.17. *There exists a unique class* $\operatorname{ch}(\mathcal{F},\sigma)$ *on* X *such that* $\gamma(\mathcal{F}) = \widetilde{\sigma}^*\operatorname{ch}(\mathcal{F},\sigma)$.

PROOF. We have

$$\begin{aligned} i_E^*\gamma(\mathcal{F}) &= i_E^*\Bigl(\sum_{j\geq 0}(-1)^j\bigl(\operatorname{Tor}_j(\mathcal{F},\sigma)\bigr)\Bigr)\\ &= \sum_{j\geq 0}(-1)^j\operatorname{ch}\bigl(i_E^\dagger\left[\operatorname{Tor}_j(\mathcal{F},\sigma)\right]\bigr) && \text{by induction property (ii)}\\ &= \operatorname{ch}\bigl(i_E^\dagger\widetilde{\sigma}^\dagger[\mathcal{F}]\bigr) = \operatorname{ch}\bigl(q^\dagger i_Y^\dagger[\mathcal{F}]\bigr) = q^*\operatorname{ch}\bigl(i_Y^\dagger[\mathcal{F}]\bigr) && \text{by } (\mathrm{F}_{n-1}). \end{aligned}$$

By Proposition 3.5 (vi), there exists a unique class $\operatorname{ch}(\mathcal{F},\sigma)$ on X such that $\gamma(\mathcal{F}) = \widetilde{\sigma}^*\operatorname{ch}(\mathcal{F},\sigma)$.
□

Putting Lemma 4.16 and Lemma 4.17 together

$$\sigma^*\operatorname{ch}(\mathcal{F},\sigma) = \sigma_1^*\gamma(\mathcal{F}) = \sum_{i\geq 0}(-1)^i\operatorname{ch}\bigl(\operatorname{Tor}_i(\mathcal{F},\sigma)\bigr).$$

LEMMA 4.18. *The class* $\operatorname{ch}(\mathcal{F},\sigma)$ *does not depend on* σ.

PROOF. As usual, if we have two resolutions, we dominate them by a third one, as shown in the diagram

$$\begin{tikzcd} & \widetilde{X} \arrow[dl, "\sigma_1"'] \arrow[dd, "\sigma"] \arrow[dr, "\sigma'_1"] & \\ \widetilde{X}_1 \arrow[dr, "\widetilde{\sigma}"'] & & \widetilde{X}'_1 \arrow[dl, "\widetilde{\sigma}'"] \\ & X & \end{tikzcd}$$

Now

$$\begin{aligned}\sigma^* \operatorname{ch}(\mathcal{F}, \widetilde{\sigma}) &= \sigma_1^{*}\gamma(\mathcal{F}) && \text{by Lemma 4.17}\\ &= \sum_{i\geq 0}(-1)^i \operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F},\sigma)\big) && \text{by Lemma 4.16.}\end{aligned}$$

By symmetry $\sigma^* \operatorname{ch}(\mathcal{F}, \widetilde{\sigma}) = \sigma^* \operatorname{ch}(\mathcal{F}, \widetilde{\sigma}')$ and we get the result. □

We proved the existence statement and part (i) of Theorem 4.14 if σ consists of at most d blowups. The general case follows using the diagram above.

We must now prove Theorem 4.14 (ii). Let Y be a smooth submanifold of X. We choose $\sigma\colon \widetilde{X} \longrightarrow X$ such that $\sigma^*\mathcal{F}$ admits a locally free quotient of maximal rank and $\sigma^{-1}(Y)$ is a simple normal crossing divisor with branches D_j. We choose as usual j such that q_j^* is injective, q_j being defined by the diagram

$$\begin{tikzcd} D_j \arrow[r, "i_{D_j}"] \arrow[d, "q_j"'] & \widetilde{X} \arrow[d, "\sigma"] \\ Y \arrow[r, "i_Y"] & X \end{tikzcd}$$

We have

$$\begin{aligned}q_j^* \operatorname{ch}\big(i_Y^\dagger[\mathcal{F}]\big) &= \operatorname{ch}\big(q_j^\dagger i_Y^\dagger[\mathcal{F}]\big) = \operatorname{ch}\big(i_{D_j}^\dagger \sigma^\dagger[\mathcal{F}]\big)\\ &= \sum_{i\geq 0}(-1)^i\ i_{D_j}^* \operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F},\sigma)\big) && \text{by Proposition 4.13 (ii).}\end{aligned}$$

Now, by the point (i), we have $\sum_{i\geq 0}(-1)^i \operatorname{ch}\big(\operatorname{Tor}_i(\mathcal{F},\sigma)\big) = \sigma^* \operatorname{ch}(\mathcal{F})$. Thus we get

$$q_j^* \operatorname{ch}(i_Y^\dagger[\mathcal{F}]) = i_{D_j}^* \sigma^* \operatorname{ch}(\mathcal{F}) = q_j^*\big(i_Y^* \operatorname{ch}(\mathcal{F})\big).$$

Therefore $\operatorname{ch}\big(i_Y^\dagger[\mathcal{F}]\big) = i_Y^* \operatorname{ch}(\mathcal{F})$ and the proof is complete. □

5. The Whitney formula

In the previous section, we achieved an important step in the induction process by defining the classes $\operatorname{ch}(\mathcal{F})$ when $\mathcal{F}$ is any coherent sheaf on a n-dimensional manifold. To conclude the proof of Theorem 4.1, it remains to check properties (W_n), (F_n) and (P_n). The crux of the proof is in fact property (W_n). The main result of this section is Theorem 5.1. The other induction hypotheses will be proved in Theorem 5.14.

THEOREM 5.1. (W_n) *holds.*

To prove Theorem 5.1, we need several reduction steps.

5.1. Reduction to the case where $\mathcal{F}$ and $\mathcal{G}$ are locally free and $\mathcal{H}$ is a torsion sheaf.

PROPOSITION 5.2. *Suppose that* (W_n) *holds when* $\mathcal{F}$ *and* $\mathcal{G}$ *are locally free sheaves and* $\mathcal{H}$ *is a torsion sheaf. Then* (W_n) *holds for arbitrary sheaves.*

PROOF. We proceed by successive reductions.

LEMMA 5.3. *It is sufficient to prove* (W_n) *when* $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ *admit a locally free quotient of maximal rank.*

PROOF. We take a general exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$
Let $\sigma\colon \widetilde{X} \longrightarrow X$ be a bimeromorphic morphism such that $\sigma^*\mathcal{F}$, $\sigma^*\mathcal{G}$ and $\sigma^*\mathcal{H}$ admit locally free quotients of maximal rank (we know that such a σ exists by Theorem 4.10). We have an exact sequence defining $\mathcal{Q}$ and $\mathcal{T}_1$:

$$\begin{tikzcd}
 & & & & 0 \arrow[dr] & & 0 & & \\
 & & & & & \mathcal{Q} \arrow[ur] \arrow[dr] & & & \\
\cdots \arrow[r] & \operatorname{Tor}_1(\mathcal{G},\sigma) \arrow[r] & \operatorname{Tor}_1(\mathcal{H},\sigma) \arrow[rr] \arrow[dr] & & \sigma^*\mathcal{F} \arrow[rr] \arrow[ur] & & \sigma^*\mathcal{G} \arrow[r] & \sigma^*\mathcal{H} \arrow[r] & 0 \\
 & & & \mathcal{T}_1 \arrow[ur] \arrow[dr] & & & & & \\
 & & 0 \arrow[ur] & & 0 & & & &
\end{tikzcd}$$

Remark that $\mathcal{T}_1$ is a torsion sheaf. By Proposition 4.12 (iii), $\mathcal{Q}$ admits a locally free quotient of maximal rank and $\mathrm{ch}(\sigma^*\mathcal{F}) = \mathrm{ch}(\mathcal{T}_1) + \mathrm{ch}(\mathcal{Q})$. Furthermore,

$$[\mathcal{T}_1] - \big[\operatorname{Tor}_1(\mathcal{H},\sigma)\big] + \big[\operatorname{Tor}_1(\mathcal{G},\sigma)\big] - \cdots = 0 \text{ in } K_{\mathrm{tors}}(\widetilde{X}).$$

Then by Proposition 4.9 (i) and Proposition 4.12 (ii),

$$\begin{aligned}
\sigma^*\big(\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G})\big) &= \sum_{i\geq 0}(-1)^i\Big[\mathrm{ch}\big(\operatorname{Tor}_i(\mathcal{F},\sigma)\big) + \mathrm{ch}\big(\operatorname{Tor}_i(\mathcal{H},\sigma)\big) - \mathrm{ch}\big(\operatorname{Tor}_i(\mathcal{G},\sigma)\big)\Big] \\
&= \mathrm{ch}\big(\sigma^*\mathcal{F}\big) + \mathrm{ch}\big(\sigma^*\mathcal{H}\big) - \mathrm{ch}\big(\sigma^*\mathcal{G}\big) - \mathrm{ch}(\mathcal{T}_1) \\
&= \mathrm{ch}(\mathcal{Q}) + \mathrm{ch}\big(\sigma^*\mathcal{H}\big) - \mathrm{ch}\big(\sigma^*\mathcal{G}\big).
\end{aligned}$$

Since σ^* is injective, Lemma 5.3 is proved. □

LEMMA 5.4. *It is sufficient to prove* (W_n) *when $\mathcal{F}$, $\mathcal{G}$ admit a locally free quotient of maximal rank and $\mathcal{H}$ is a torsion sheaf.*

PROOF. By Lemma 5.3, we can assume that $\mathcal{F}, \mathcal{G}, \mathcal{H}$ admit a locally free quotient of maximal rank. In the sequel, the letter "$\mathcal{T}$" will denote a torsion sheaf and the letter "$\mathcal{E}$" a locally free sheaf. Let $\mathcal{E}_1$ be the locally free quotient of maximal rank of $\mathcal{H}$, so we have an exact sequence

$$0 \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{H} \longrightarrow \mathcal{E}_1 \longrightarrow 0.$$

We define $\mathcal{F}_1$ by the exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}_1 \longrightarrow 0.$$

Then we get a third exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{T}_1 \longrightarrow 0.$$

We have by definition $\mathrm{ch}(\mathcal{H}) = \overline{\mathrm{ch}}(\mathcal{E}_1) + \mathrm{ch}(\mathcal{T}_1)$. Thus,

$$\begin{aligned}
\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G}) &= \big(\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{T}_1) - \mathrm{ch}(\mathcal{F}_1)\big) + \big(\mathrm{ch}(\mathcal{F}_1) + \overline{\mathrm{ch}}(\mathcal{E}_1) - \mathrm{ch}(\mathcal{G})\big) \\
&\qquad - \big(\mathrm{ch}(\mathcal{T}_1) + \overline{\mathrm{ch}}(\mathcal{E}_1) - \mathrm{ch}(\mathcal{H})\big) \\
&= \big(\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{T}_1) - \mathrm{ch}(\mathcal{F}_1)\big) + \big(\mathrm{ch}(\mathcal{F}_1) + \overline{\mathrm{ch}}(\mathcal{E}_1) - \mathrm{ch}(\mathcal{G})\big).
\end{aligned}$$

Let $\mathcal{E}_2$ be the locally free quotient of maximal rank of $\mathcal{G}$. We define $\mathcal{T}_2$ by the exact sequence

$$0 \longrightarrow \mathcal{T}_2 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}_2 \longrightarrow 0.$$

The morphism from $\mathcal{G}$ to $\mathcal{E}_1$ (via $\mathcal{H}$) induces a morphism $\mathcal{E}_2 \longrightarrow \mathcal{E}_1$ which remains of course surjective. Let $\mathcal{E}_3$ be the kernel of this morphism, then $\mathcal{E}_3$ is a locally free sheaf. We get an exact sequence

$$0 \longrightarrow \mathcal{T}_2 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{E}_3 \longrightarrow 0.$$

Therefore $\mathcal{F}_1$ admits a locally free quotient of maximal rank and $\mathrm{ch}(\mathcal{F}_1) = \mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_3)$. On the other hand, by Proposition 4.12 (i), $\overline{\mathrm{ch}}\big(\mathcal{E}_1\big) + \overline{\mathrm{ch}}\big(\mathcal{E}_3\big) = \overline{\mathrm{ch}}\big(\mathcal{E}_2\big)$, and we obtain $\mathrm{ch}\big(\mathcal{F}_1\big) + \overline{\mathrm{ch}}\big(\mathcal{E}_1\big) - \mathrm{ch}(\mathcal{G}) = \big(\mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_3)\big) + \big(\overline{\mathrm{ch}}(\mathcal{E}_2) - \overline{\mathrm{ch}}(\mathcal{E}_3)\big) - \big(\mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_2)\big) = 0$. Therefore,

$$\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G}) = \mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{T}_1) - \mathrm{ch}(\mathcal{F}_1).$$

Since $\mathcal{T}_1$ is a torsion sheaf, we are done. □

We can now conclude the proof of Proposition 5.2.

By Lemma 5.4, we can suppose that $\mathcal{F}$, $\mathcal{G}$ admit locally free quotients of maximal rank and $\mathcal{H}$ is a torsion sheaf. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be the locally free quotients of maximal rank of $\mathcal{F}$ and $\mathcal{G}$. We define $\mathcal{T}_1$ and $\mathcal{T}_2$ by the two exact sequences

$$0 \longrightarrow \mathcal{T}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E}_1 \longrightarrow 0$$

$$0 \longrightarrow \mathcal{T}_2 \longrightarrow \mathcal{G} \longrightarrow \mathcal{E}_2 \longrightarrow 0.$$

The morphism $\mathcal{F} \longrightarrow \mathcal{G}$ induces a morphism $\mathcal{T}_1 \longrightarrow \mathcal{T}_2$. We get a morphism $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$ with torsion kernel and cokernel. Since $\mathcal{E}_1$ is a locally free sheaf, this morphism is injective. In the following diagram, we introduce the cokernels $\mathcal{T}_3$ and $\mathcal{T}_4$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathcal{T}_1 \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E}_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{T}_2 \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & \mathcal{E}_2 \arrow[r] \arrow[d] & 0 \\
 & \mathcal{T}_3 \arrow[d] & \mathcal{H} \arrow[d] & \mathcal{T}_4 \arrow[d] & \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

By the nine lemma, $0 \longrightarrow \mathcal{T}_3 \longrightarrow \mathcal{H} \longrightarrow \mathcal{T}_4 \longrightarrow 0$ is an exact sequence of torsion sheaves. Then by Proposition 4.2 (i),

$$\begin{aligned}
\mathrm{ch}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \mathrm{ch}(\mathcal{G}) &= \mathrm{ch}\big(\mathcal{T}_1\big) + \overline{\mathrm{ch}}\big(\mathcal{E}_1\big) + \mathrm{ch}\big(\mathcal{T}_3\big) + \mathrm{ch}\big(\mathcal{T}_4\big) - \mathrm{ch}\big(\mathcal{T}_2\big) - \overline{\mathrm{ch}}\big(\mathcal{E}_2\big) \\
&= \overline{\mathrm{ch}}\big(\mathcal{E}_1\big) + \mathrm{ch}\big(\mathcal{T}_4\big) - \overline{\mathrm{ch}}\big(\mathcal{E}_2\big).
\end{aligned}$$

This finishes the proof. □

5.2. A structure theorem for coherent torsion sheaves of projective dimension one. In section 5.1 we have reduced the Whitney formula to the particular case where $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is a torsion sheaf. We are now going to prove that it is sufficient to suppose that $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersurface of X. The main tool of this section is the following proposition:

Proposition 5.5. *Let $\mathcal{H}$ be a torsion sheaf which admits a global locally free resolution of length two. Then there exist a bimeromorphic morphism $\sigma : \widetilde{X} \longrightarrow X$ obtained by a finite number of blowups with smooth centers, a simple normal crossing divisor D, $D \subseteq X$, and an increasing sequence $\big(D_i\big)_{1\leq i\leq r}$ of subdivisors of D such that $\sigma^*\mathcal{H}$ is everywhere locally isomorphic to $\bigoplus_{i=1}^{r} \mathcal{O}_{\widetilde{X}}/\mathcal{I}_{D_i}$.*

Proof. Let $0 \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{H} \longrightarrow 0$ be a locally free resolution of $\mathcal{H}$, such that $\operatorname{rank}\big(\mathcal{E}_1\big) = \operatorname{rank}\big(\mathcal{E}_2\big) = r$. Recall that the kth Fitting ideal of $\mathcal{H}$ is the coherent ideal sheaf generated by the determinants of all the $k\times k$ minors of M when M is any local matrix realization in coordinates of the morphism $\mathcal{E}_1 \longrightarrow \mathcal{E}_2$ (for a general presentation of the Fitting ideals, see [**Ei**]). We have

$$\operatorname{Fitt}_1(\mathcal{H}) \supseteq \operatorname{Fitt}_2(\mathcal{H}) \supseteq \cdots \supseteq \operatorname{Fitt}_r(\mathcal{H}) \supsetneq \{0\}.$$

These ideals have good functoriality properties. Indeed, if $\sigma : \widetilde{X} \longrightarrow X$ is a bimeromorphic morphism, the sequence $0 \longrightarrow \sigma^*\mathcal{E}_1 \longrightarrow \sigma^*\mathcal{E}_2 \longrightarrow \sigma^*\mathcal{H} \longrightarrow 0$ is exact and $\operatorname{Fitt}_j\big(\sigma^*\mathcal{H}\big) = \sigma^*\operatorname{Fitt}_j(\mathcal{H})$ (by $\sigma^*\operatorname{Fitt}_j(\mathcal{H})$, we mean of course its image in $\mathcal{O}_{\widetilde{X}}$). By the Hironaka theorem, we can suppose, after taking a finite number of pullbacks under blowups with smooth centers, that all the Fitting ideals $\operatorname{Fitt}_k(\mathcal{F})$ are ideal sheaves associated with effective normal crossing divisors D'_k. Now, take an element x of X. Consider an exact sequence

$$\mathcal{O}_U^r \xrightarrow[M]{} \mathcal{O}_U^r \longrightarrow \mathcal{H}_{|U} \longrightarrow 0$$

in a neighbourhood of x. The matrix M is a $r\times r$ matrix of holomorphic functions on U. Let $\big\{\phi_1 = 0\big\}$ be an equation of D_1 around x. Then we can write $M = \phi_1 M_1$ and the coefficients of M_1 generate $\mathcal{O}_x$. Thus, at least one of these coefficients does not vanish at x. We can suppose that it is the upper-left one. By Gauss elimination process, we can assume

$$M = \phi_1 \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \boxed{\quad M_2 \quad} & \\ 0 & & & \end{pmatrix}.$$

Then, since $\operatorname{Fitt}_k(\mathcal{F})_{|U} = \operatorname{Fitt}_k(M)$, we get $\operatorname{Fitt}_2(M) = \phi_1^2 \operatorname{Fitt}_1(M_2)$. Since $\operatorname{Fitt}_2(M)$ is principal, so is $\operatorname{Fitt}_1(M_2)$ and we write $\operatorname{Fitt}_1(M_2) = \big(\phi_2\big)$. Then, by the same argument as above, we can assume

$$M = \begin{pmatrix} \phi_1 & 0 & \cdots & \cdots & 0 \\ 0 & \phi_1\phi_2 & 0 & \cdots & 0 \\ \vdots & 0 & & & \\ \vdots & \vdots & & \boxed{\quad M_3 \quad} & \\ 0 & 0 & & & \end{pmatrix}.$$

By this algorithm, we get

$$M = \begin{pmatrix} \phi_1 & 0 & \cdots & 0 \\ 0 & \phi_1\phi_2 & & \\ \vdots & & \ddots & \\ 0 & & & \phi_1\cdots\phi_r \end{pmatrix}$$

and then $\mathcal{F}_{|U} \simeq \left(\mathcal{O}_X \big/ \phi_1 \mathcal{O}_X\right)_{|U} \oplus \cdots \oplus \left(\mathcal{O}_X \big/ \phi_1 \ldots \phi_r \mathcal{O}_X\right)_{|U}$. Thus, if $D_1, \ldots, D_r$ are the divisors of $\phi_1, \phi_1\,\phi_2, \ldots, \phi_1\,\phi_2 \ldots \phi_r$, we have $D_k = D'_k - D'_{k-1}$, which shows that the divisors D_k are intrinsically defined by $\mathcal{F}$. □

From now on, we will say that a torsion sheaf $\mathcal{H}$ is *principal* if it is everywhere locally isomorphic to a fixed sheaf $\bigoplus_{i=1}^{r} \mathcal{O}_X \big/ \mathcal{I}_{D_i}$ where the D_i are (non necessarily reduced) effective normal crossing divisors and $D_1 \leq D_2 \leq \cdots \leq D_r$. We will denote by $\nu(\mathcal{H})$ the number of branches of D, counted with their multiplicities.

Proposition 5.6. *It suffices to prove the Whitney formula when $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersuface.*

Proof. We proceed in several steps.

Lemma 5.7. *Consider an exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow i_{Y*}\mathcal{E} \longrightarrow 0$ where Y is a smooth hypersurface of X, $\mathcal{G}$ is a locally free sheaf on X and $\mathcal{E}$ is a locally free sheaf on Y. Then $\mathcal{F}$ is locally free on X.*

Proof. Let m_x be the maximal ideal of the local ring $\mathcal{O}_x$. By Nakayama's lemma, it suffices to show that for every x in X, $\mathrm{Tor}_1^{\mathcal{O}_x}\big(\mathcal{F}_x, \mathcal{O}_x/m_x\big) = 0$. Since Y is a hypersurface, $i_{Y*}\mathcal{E}$ admits a locally free resolution of length two. Thus $\mathrm{Tor}_1^{\mathcal{O}_x}\big(\mathcal{F}_x, \mathcal{O}_x/m_x\big) \simeq \mathrm{Tor}_2^{\mathcal{O}_x}\big((i_{Y*}\mathcal{E})_x, \mathcal{O}_x/m_x\big) = 0$. □

Lemma 5.8. *It suffices to prove* (W_n) *when $\mathcal{F}$, $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is principal.*

Proof. By Proposition 5.2, it is enough to prove the Whitney formula when $\mathcal{F}$, $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is a torsion sheaf. So we suppose that $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ verify these hypotheses. By Proposition 5.5, there exists a bimeromorphic morphism $\sigma : \widetilde{X} \longrightarrow X$ such that $\sigma^*\mathcal{H}$ is principal. We have an exact sequence

$$0 \longrightarrow \mathrm{Tor}_1(\mathcal{H}, \sigma) \longrightarrow \sigma^*\mathcal{F} \longrightarrow \sigma^*\mathcal{G} \longrightarrow \sigma^*\mathcal{H} \longrightarrow 0.$$

But $\mathrm{Tor}_1(\mathcal{H}, \sigma)$ is a torsion sheaf and $\sigma^*\mathcal{F}$ is locally free, so we get an exact sequence

$$0 \longrightarrow \sigma^*\mathcal{F} \longrightarrow \sigma^*\mathcal{G} \longrightarrow \sigma^*\mathcal{H} \longrightarrow 0.$$

and we have $\mathrm{Tor}_i(\mathcal{H}, \sigma) = 0$ for $i \geq 1$. By Proposition 3.7 (ii), we obtain the equalities $\overline{\mathrm{ch}}(\sigma^*\mathcal{F}) = \sigma^*\overline{\mathrm{ch}}(\mathcal{F})$ and $\overline{\mathrm{ch}}(\sigma^*\mathcal{G}) = \sigma^*\overline{\mathrm{ch}}(\mathcal{G})$, and by Proposition 4.2 (iii) we get $\mathrm{ch}(\sigma^*\mathcal{H}) = \sigma^*\,\mathrm{ch}(\mathcal{H})$. Thus

$$\sigma^*\big(\overline{\mathrm{ch}}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}) - \overline{\mathrm{ch}}(\mathcal{G})\big) = \overline{\mathrm{ch}}(\sigma^*\mathcal{F}) + \mathrm{ch}(\sigma^*\mathcal{H}) - \overline{\mathrm{ch}}(\sigma^*\mathcal{G}).$$

□

Lemma 5.9. *Suppose that* (W_n) *holds if $\mathcal{F}$, $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersurface. Then* (W_n) *holds when $\mathcal{F}$, $\mathcal{G}$ are locally free sheaves and $\mathcal{H}$ is principal.*

Proof. We argue by induction on $\nu(\mathcal{H})$. If $\nu(\mathcal{H}) = 0$, $\mathcal{H} = 0$ and $\mathcal{F} \simeq \mathcal{G}$. If $\nu(\mathcal{H}) = 1$, $\mathcal{H}$ is the push-forward of a locally free sheaf on a smooth hypersurface and there is nothing to prove. In the general case, let Y be a branch of D_1. Since $Y \leq D_i$ for every i with $1 \leq i \leq r$, we can see that $\mathcal{E} = \mathcal{H}_{|Y}$ is locally free on Y. Besides, if we define $\widetilde{\mathcal{H}}$ by the exact sequence

$$0 \longrightarrow \widetilde{\mathcal{H}} \longrightarrow \mathcal{H} \longrightarrow i_{Y*}\mathcal{E} \longrightarrow 0,$$

$\widetilde{\mathcal{H}}$ is everywhere locally isomorphic to $\bigoplus_{i=1}^{r} \mathcal{O}_X/_{\mathcal{I}_{D_i - Y}}$. Thus $\widetilde{\mathcal{H}}$ is principal and $\nu(\widetilde{\mathcal{H}}) = \nu(\mathcal{H}) - 1$. We define $\widetilde{\mathcal{E}}$ by the exact sequence:

$$0 \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \mathcal{G} \longrightarrow i_{Y*}\mathcal{E} \longrightarrow 0.$$

By Lemma 5.7, $\widetilde{\mathcal{E}}$ is locally free. Furthermore, we have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \widetilde{\mathcal{E}} \longrightarrow \widetilde{\mathcal{H}} \longrightarrow 0.$$

By induction, $\overline{\mathrm{ch}}\big(\widetilde{\mathcal{E}}\big) = \overline{\mathrm{ch}}(\mathcal{F}) + \mathrm{ch}\big(\widetilde{\mathcal{H}}\big)$ and by our hypothesis $\mathrm{ch}(\mathcal{G}) = \overline{\mathrm{ch}}\big(\widetilde{\mathcal{E}}\big) + \mathrm{ch}\big(i_{Y*}\mathcal{E}\big)$. Since $\widetilde{\mathcal{H}}$, $\mathcal{H}$ and $i_{Y*}\mathcal{E}$ are torsion sheaves, $\mathrm{ch}(\mathcal{H}) = \mathrm{ch}\big(\widetilde{\mathcal{H}}\big) + \mathrm{ch}\big(i_{Y*}\mathcal{E}\big)$ and we get $\overline{\mathrm{ch}}(\mathcal{G}) = \overline{\mathrm{ch}}(\mathcal{F}) + \mathrm{ch}(\mathcal{H})$. □

Putting the two lemmas together, we obtain Proposition 5.6. □

5.3. Proof of the Whitney formula. We are now ready to prove Theorem 5.1.

In the sections 5.1 and 5.2, we have made successive reductions in order to prove the Whitney formula in a tractable context, so that we are reduced to the case where $\mathcal{F}$ and $\mathcal{G}$ are locally free sheaves and $\mathcal{H} = i_{Y*}\mathcal{E}$, where Y is a smooth hypersurface of X and $\mathcal{E}$ is a locally free sheaf on Y. Our working hypotheses will be these.

Let us briefly explain the sketch of the argument. We consider the sheaf $\widetilde{\mathcal{G}}$ on $X \times \mathbb{P}^1$ obtained by deformation of the extension class of the exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

Then $\widetilde{\mathcal{G}}_{|X\times\{0\}} \simeq \mathcal{F} \oplus \mathcal{H}$ and $\widetilde{\mathcal{G}}_{|X\times\{t\}} \simeq \mathcal{G}$ for $t \neq 0$. It will turn out that $\widetilde{\mathcal{G}}$ admits a locally free quotient of maximal rank $\mathcal{Q}$ on the blowup of $X \times \mathbb{P}^1$ along $Y \times \{0\}$, and the associated kernel $\mathcal{N}$ will be the push-forward of a locally free sheaf on the exceptional divisor E, say $\mathcal{N} = i_{E*}\mathcal{L}$. Then we consider the class $\alpha = \overline{\mathrm{ch}}(\mathcal{Q}) + i_{E*}\big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}(N_{E/X})^{-1}\big)$ on the blowup. After explicit computations, it will appear that α is the pullback of a form β on the base $X \times \mathbb{P}^1$. By the $\mathbb{P}^1$-homotopy invariance of Deligne cohomology (Proposition 3.3 (vi)), $\beta_{|X\times\{t\}}$ does not depend on t. This will give the desired result.

Let us first introduce some notations. The morphism $\mathcal{F} \longrightarrow \mathcal{G}$ will be denoted by γ. Let s be a global section of $\mathcal{O}_{\mathbb{P}^1}(1)$ which vanishes exactly at $\{0\}$. Let $\mathrm{pr}_1 : X \times \mathbb{P}^1 \longrightarrow X$ be the projection on the first factor. The relative $\mathcal{O}(1)$, namely $\mathcal{O}_X \boxtimes \mathcal{O}_{\mathbb{P}^1}(1)$, will still be denoted by $\mathcal{O}(1)$. We define a sheaf $\widetilde{\mathcal{G}}$ on $X \times \mathbb{P}^1$ by the exact sequence

$$0 \longrightarrow \mathrm{pr}_1^*\,\mathcal{F} \xrightarrow[(\mathrm{id}\otimes s,\,\gamma)]{} \mathrm{pr}_1^*\,\mathcal{F}(1) \oplus \mathrm{pr}_1^*\,\mathcal{G} \longrightarrow \widetilde{\mathcal{G}} \longrightarrow 0.$$

Remark that $\widetilde{\mathcal{G}}_0 \simeq \mathcal{F} \oplus \mathcal{H}$ and $\widetilde{\mathcal{G}}_t \simeq \mathcal{G}$ if $t \neq 0$.

Lemma 5.10. *There exist two exact sequences*

(i) $$0 \longrightarrow \mathrm{pr}_1^*\,\mathcal{F}(1) \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{pr}_1^*\,\mathcal{H} \longrightarrow 0$$

(ii) $$0 \longrightarrow \widetilde{\mathcal{G}} \longrightarrow \mathrm{pr}_1^*\,\mathcal{G}(1) \longrightarrow i_{X_0*}\mathcal{H} \longrightarrow 0.$$

Remark 5.11. (i) implies that $\widetilde{\mathcal{G}}$ is flat over $\mathbb{P}^1$.

PROOF. (i) The morphism $\mathrm{pr}_1^*\mathcal{F}(1)\oplus\mathrm{pr}_1^*\mathcal{G}\twoheadrightarrow\mathrm{pr}_1^*\mathcal{G}\twoheadrightarrow\mathrm{pr}_1^*\mathcal{H}$ induces a morphism $\widetilde{\mathcal{G}}\twoheadrightarrow\mathrm{pr}_1^*\mathcal{H}$. If $\mathcal{K}$ is the kernel of this morphism, we obtain an exact sequence

$$0\longrightarrow\mathrm{pr}_1^*\mathcal{F}\xrightarrow[(\mathrm{id}\otimes s,\,\mathrm{id})]{}\mathrm{pr}_1^*\mathcal{F}(1)\oplus\mathrm{pr}_1^*\mathcal{F}\longrightarrow\mathcal{K}\longrightarrow 0.$$

Thus $\mathcal{K}=\mathrm{pr}_1^*\mathcal{F}(1)$.

(ii) We consider the morphism $\mathrm{pr}_1^*\mathcal{F}(1)\oplus\mathrm{pr}_1^*\mathcal{G}\twoheadrightarrow\mathrm{pr}_1^*\mathcal{G}(1)$

$$f+g\longrightarrow\gamma(f)-g\otimes s.$$

It induces a morphism $\phi\colon\widetilde{\mathcal{G}}\longrightarrow\mathrm{pr}_1^*\mathcal{G}(1)$. The last morphism of (ii) is just the composition

$$\mathrm{pr}_1^*\mathcal{G}(1)\twoheadrightarrow i_{X_0*}\mathcal{G}\twoheadrightarrow i_{X_0*}\mathcal{H}.$$

The cokernel of this morphism has support in $X\times\{0\}$. Besides, the action of t on this cokernel is zero. The restriction of ϕ to the fiber $X_0=X\times\{0\}$ is the morphism $\mathcal{F}\oplus\mathcal{H}\longrightarrow\mathcal{G}$, thus the sequence $\widetilde{\mathcal{G}}\longrightarrow\mathrm{pr}_1^*\mathcal{G}(1)\longrightarrow i_{X_0*}\mathcal{H}\longrightarrow 0$ is exact. The kernel of ϕ, as its cokernel, is an $\mathcal{O}_{X_0}$-module. Thus we can find $\mathcal{Z}$ such that $\ker\phi=i_{X_0*}\mathcal{Z}$. Since X_0 is a hypersurface of $X\times\mathbb{P}^1$, for every coherent sheaf $\mathcal{L}$ on $X\times\mathbb{P}^1$, we have $\mathrm{Tor}_2(\mathcal{L},i_{X_0})=0$. Applying this to $\mathcal{L}=\widetilde{\mathcal{G}}/_{i_{X_0*}\mathcal{Z}}$ and using Remark 5.11, we get

$$\mathrm{Tor}_1(i_{X_0*}\mathcal{Z},i_{X_0})\subseteq\mathrm{Tor}_1(\widetilde{\mathcal{G}},i_{X_0})=\{0\}.$$

But $\mathrm{Tor}_1(i_{X_0*}\mathcal{Z},i_{X_0})\simeq\mathcal{Z}\otimes N^*_{X_0/X\times\mathbb{P}^1}\simeq\mathcal{Z}$, so $\mathcal{Z}=\{0\}$. □

Recall now that $\mathcal{H}=i_{Y*}\mathcal{E}$ where Y is a smooth hypersurface of X and $\mathcal{E}$ is a locally free sheaf on Y. We consider the space $M_{Y/X}$ of the deformation of the normal cone of Y in X (see [**Fu**]). Basically, $M_{Y/X}$ is the blowup of $X\times\mathbb{P}^1$ along $Y\times\{0\}$. Let $\sigma\colon M_{Y/X}\longrightarrow X\times\mathbb{P}^1$ be the canonical morphism. Then σ^*X_0 is a Cartier divisor in $M_{Y/X}$ with two simple branches: $E=\mathbb{P}(N_{Y/X}\oplus\mathcal{O}_Y)$ and $D=\mathrm{Bl}_Y X\simeq X$, which intersect at $\mathbb{P}(N_{Y/X})\simeq Y$. The projection of the blowup from E to $Y\times\{0\}$ will be denoted by q, and the canonical isomorphism from D to $X\times\{0\}$ will be denoted by μ.

We now show:

LEMMA 5.12. *The sheaf $\sigma^*\widetilde{\mathcal{G}}$ admits a locally free quotient with maximal rank on $M_{Y/X}$, and the associated kernel $\mathcal{N}$ is the push-forward of a locally free sheaf on E. More explicitly, if F is the excess conormal bundle of q, $\mathcal{N}=i_{E*}(q^*\mathcal{E}\otimes F)$.*

PROOF. We start from the exact sequence

$$0\longrightarrow\widetilde{\mathcal{G}}\longrightarrow\mathrm{pr}_1^*\mathcal{G}(1)\longrightarrow i_{X_0*}\mathcal{H}\longrightarrow 0.$$

We define $\mathcal{Q}$ by the exact sequence $0\longrightarrow\mathcal{Q}\longrightarrow\sigma^*\mathrm{pr}_1^*\mathcal{G}(1)\longrightarrow\sigma^*i_{X_0*}\mathcal{H}\longrightarrow 0$. Since $\sigma^*i_{X_0*}\mathcal{H}$ is the push-forward of a locally free sheaf on E, by Lemma 5.7, the sheaf $\mathcal{Q}$ is locally free on $M_{Y/X}$. The sequence

$$0\longrightarrow\mathrm{Tor}_1(i_{X_0*}\mathcal{H},\sigma)\longrightarrow\sigma^*\widetilde{\mathcal{G}}\longrightarrow\mathcal{Q}\longrightarrow 0$$

is exact. The first sheaf being a torsion sheaf, $\mathcal{Q}$ is a locally free quotient of $\widetilde{\mathcal{G}}$ with maximal rank. Besides, using the notations given in the following diagram

$$\begin{tikzcd} E \arrow[r, hook, "i_E"] \arrow[d, "q"'] & M_{Y/X} \arrow[d, "\sigma"] \\ Y \times \{0\} \arrow[r, hook, "i_{Y\times\{0\}}"'] & X \times \mathbb{P}^1 \end{tikzcd}$$

we have $\mathrm{Tor}_1(i_{X_0 *}\mathcal{H}, \sigma) = i_{E*}\big(q^*\mathcal{E} \otimes F\big)$ where F is the excess conormal bundle of q (see [**Bo-Se**, § 15]. Be aware of the fact that what we note F is F^* in [**Bo-Se**]). This finishes the proof. $\square$

We consider now the exact sequence $0 \longrightarrow \mathcal{N} \longrightarrow \sigma^*\widetilde{\mathcal{G}} \longrightarrow \mathcal{Q} \longrightarrow 0$ where $\mathcal{Q}$ is locally free on $M_{Y/X}$ and $\mathcal{N} = i_{E*}\big(q^*\mathcal{E} \otimes F\big) = i_{E*}\mathcal{L}$. We would like to introduce the form $\mathrm{ch}(\sigma^*\widetilde{\mathcal{G}})$, but it is not defined since $M_{Y/X}$ is of dimension $n+1$. However, $\sigma^*\widetilde{\mathcal{G}}$ fits in a short exact sequence where the Chern classes of the two other sheaves can be defined. Remark that we need Lemma 5.12 to perform this trick, since it cannot be done on $X \times \mathbb{P}^1$.

Lemma 5.13. *Let α be the Deligne class on $M_{Y/X}$ defined by*

$$\alpha = \overline{\mathrm{ch}}(\mathcal{Q}) + i_{E*}\Big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\Big).$$

(i) *The class α is the pullback of a Deligne class on $X \times \mathbb{P}^1$.*
(ii) *We have $i_D^*\alpha = \mu^*\,\mathrm{ch}\big(\widetilde{\mathcal{G}}_0\big)$.*

Proof. We compute explicitly $i_E^*\alpha$.

$$\begin{aligned} i_E^*\, i_{E*}\Big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\Big) &= \overline{\mathrm{ch}}(\mathcal{L})\;\; \mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\;\, c_1\big(N_{E/M_{Y/X}}\big) \\ &\qquad\qquad\qquad\qquad \text{(by Proposition 3.5 (vii))} \\ &= \overline{\mathrm{ch}}(\mathcal{L})\Big(1 - e^{-c_1\left(N_{E/M_{Y/X}}\right)}\Big) \\ &= \overline{\mathrm{ch}}(\mathcal{L}) - \overline{\mathrm{ch}}\Big(\mathcal{L} \otimes N^*_{E/M_{Y/X}}\Big) \\ &\qquad\qquad\qquad\qquad \text{(by Proposition 3.7 (iii))} \\ &= \overline{\mathrm{ch}}\big(i_E^*\mathcal{N}\big) - \overline{\mathrm{ch}}\Big(\mathcal{L} \otimes N^*_{E/M_{Y/X}}\Big). \end{aligned}$$

From the exact sequence $0 \longrightarrow \mathcal{N} \longrightarrow \sigma^*\widetilde{\mathcal{G}} \longrightarrow \mathcal{Q} \longrightarrow 0$, we get the exact sequence of locally free sheaves on E: $0 \longrightarrow i_E^*\mathcal{N} \longrightarrow i_E^*\;\sigma^*\widetilde{\mathcal{G}} \longrightarrow i_E^*\mathcal{Q} \longrightarrow 0$. Since $i_E^*\;\overline{\mathrm{ch}}(\mathcal{Q}) = \overline{\mathrm{ch}}\big(i_E^*\mathcal{Q}\big)$, we obtain

$$\begin{aligned} i_E^*\alpha &= \overline{\mathrm{ch}}\big(i_E^*\mathcal{Q}\big) + \overline{\mathrm{ch}}\big(i_E^*\mathcal{N}\big) - \overline{\mathrm{ch}}\big(\mathcal{L} \otimes N^*_{E/M_{Y/X}}\big) \\ &= \overline{\mathrm{ch}}\big(i_E^*\sigma^*\widetilde{\mathcal{G}}\big) - \overline{\mathrm{ch}}\big(\mathcal{L} \otimes N^*_{E/M_{Y/X}}\big) && \text{by Proposition 3.7 (i)} \\ &= \overline{\mathrm{ch}}\big(q^* i_Y^*\mathcal{F}\big) + \overline{\mathrm{ch}}\big(q^* i_Y^*\mathcal{H}\big) - \overline{\mathrm{ch}}\big(\mathcal{L} \otimes N^*_{E/M_{Y/X}}\big) \\ &= q^*\overline{\mathrm{ch}}\big(i_Y^*\mathcal{F}\big) + q^*\overline{\mathrm{ch}}(\mathcal{E}) - \overline{\mathrm{ch}}\big(q^*\mathcal{E} \otimes F \otimes N^*_{E/M_{Y/X}}\big) && \text{by Proposition 3.7 (ii).} \end{aligned}$$

Recall that the conormal excess bundle F is the line bundle defined by the exact sequence

$$0 \longrightarrow F \longrightarrow q^*N^*_{Y/X\times\mathbb{P}^1} \longrightarrow N^*_{E/M_{Y/X}} \longrightarrow 0.$$

Thus, $\det\big(q^*N^*_{Y/X\times\mathbb{P}^1}\big) = F\otimes N^*_{E/M_{Y/X}}$. Since $\det\big(q^*N^*_{Y/X\times\mathbb{P}^1}\big) = q^*\det\big(N^*_{Y/X\times\mathbb{P}^1}\big)$, we get by Proposition 3.7 (ii) again

$$i_E^*\alpha = q^*\big[\,\overline{\mathrm{ch}}\big(i_Y^*\mathcal{F}\big) + \overline{\mathrm{ch}}(\mathcal{E}) - \overline{\mathrm{ch}}\big(\mathcal{E}\otimes\det\big(N^*_{Y/X\times\mathbb{P}^1}\big)\big)\big].$$

This proves (i).

(ii) The divisors E and D meet transversally. Then, by Proposition 3.5 (iv),

$$\begin{aligned} i_D^*\alpha &= i_D^*\overline{\mathrm{ch}}(\mathcal{Q}) + i_D^* i_{E*}\Big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\Big)\\ &= \overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + i_{E\cap D\longrightarrow D*}\; i^*_{E\cap D\longrightarrow E}\Big(\overline{\mathrm{ch}}(\mathcal{L})\,\mathrm{td}\big(N_{E/M_{Y/X}}\big)^{-1}\Big)\\ &= \overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + i_{E\cap D\longrightarrow D*}\Big(\overline{\mathrm{ch}}\big(i^*_{E\cap D\longrightarrow E}\mathcal{L}\big)\,\mathrm{td}\big(N_{E\cap D/D}\big)^{-1}\Big). \end{aligned}$$

We remark now that $i^*_{E\cap D\longrightarrow E}\mathcal{L} = i^*_{D\cap E}\,\mathcal{N}$. Since $\dim(E\cap D) = n-1$, we obtain

$$i_D^*\alpha = \overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + \mathrm{ch}\big(i_{E\cap D\longrightarrow D*}\, i^*_{E\cap D}\mathcal{N}\big) = \overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + \mathrm{ch}\big(i_D^*\mathcal{N}\big).$$

We have the exact sequence on D: $0\longrightarrow i_D^*\mathcal{N}\longrightarrow i_D^*\,\sigma^*\widetilde{\mathcal{G}}\longrightarrow i_D^*\mathcal{Q}\longrightarrow 0$. Therefore $i_D^*\,\sigma^*\widetilde{\mathcal{G}}$ admits a locally free quotient of maximal rank and, μ being an isomorphism,

$$\overline{\mathrm{ch}}\big(i_D^*\mathcal{Q}\big) + \mathrm{ch}(i_D^*\mathcal{N}) = \mathrm{ch}\big(i_D^*\,\sigma^*\widetilde{\mathcal{G}}\big) = \mathrm{ch}\big(\mu^*\widetilde{\mathcal{G}}_0\big) = \mu^*\,\mathrm{ch}\big(\widetilde{\mathcal{G}}_0\big).$$

□

We are now ready to use the homotopy property for Deligne cohomology (Proposition 3.3 (vi)). Let α be the form defined in Lemma 5.13. Using (i) of this lemma and Proposition 3.5 (vi), we can write $\alpha = \sigma^*\beta$. Thus $i_D^*\,\alpha = i_D^*\,\sigma^*\beta = \mu^* i_{X_0}^*\beta$. By (ii) of the same lemma, $i_D^*\alpha = \mu^*\,\mathrm{ch}\big(\widetilde{\mathcal{G}}_0\big)$ and we get $i_{X_0}^*\beta = \mathrm{ch}\big(\widetilde{\mathcal{G}}_0\big)$. If $t\in\mathbb{P}^1\backslash\{0\}$, we have clearly $\beta_{|X_t} = \overline{\mathrm{ch}}(\mathcal{G})$. Since $\beta_{|X_t} = \beta_{|X_0}$ we obtain

$$\overline{\mathrm{ch}}(\mathcal{G}) = \mathrm{ch}\big(\widetilde{\mathcal{G}}_0\big) = \overline{\mathrm{ch}}(\mathcal{F}) + \mathrm{ch}(\mathcal{H}).$$

□

We can now establish the remaining induction properties.

THEOREM 5.14. *The following assertions are valid:*

(i) *Property* (F_n) *holds.*
(ii) *Property* (P_n) *holds.*

PROOF. (i) We take $y = [\mathcal{F}]$. Let us first suppose that f is a bimeromorphic map. Then there exists a bimeromorphic map $\sigma\colon\widetilde{X}\longrightarrow X$ such that $(f\circ\sigma)^*\mathcal{F}$ admits a locally free quotient of maximal rank. Then by Theorem 4.14 (i),

$$\sigma^*\,\mathrm{ch}\big(f^{\dagger}[\mathcal{F}]\big) = \mathrm{ch}\big(\sigma^{\dagger}\,f^{\dagger}[\mathcal{F}]\big) = (f\circ\sigma)^*\,\mathrm{ch}\,\mathcal{F} = \sigma^*\big[f^*\,\mathrm{ch}(\mathcal{F})\big].$$

Suppose now that f is a surjective map. Then there exist two bimeromorphic maps π_X : $\widetilde{X}\longrightarrow X$, $\pi_Y\colon\widetilde{Y}\longrightarrow Y$ and a surjective map $\widetilde{f}\colon\widetilde{X}\longrightarrow\widetilde{Y}$ such that:

– the diagram
$$\begin{tikzcd} \widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_X"'] & \widetilde{Y} \arrow[d, "\pi_Y"'] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
is commutative.

– the sheaf $\pi_Y^* \mathcal{F}$ admits a locally free quotient $\mathcal{E}$ of maximal rank.

We can write $\pi_Y^\dagger[\mathcal{F}] = [\mathcal{E}] + \widetilde{y}$ in $K(\widetilde{Y})$, where $\widetilde{y}$ is in the image of the natural map $\iota : K_{\mathrm{tors}}(\widetilde{Y}) \longrightarrow K(\widetilde{Y})$. The functoriality property being known for bimeromorphic maps, it holds for π_X and π_Y. The result is now a consequence of Proposition 3.7 (ii) and Proposition 4.2 (iii).

In the general case, we consider the diagram used in the proof of Proposition 4.2 (iii)

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\pi_X"'] & W \arrow[r, "i_W"] \arrow[d, "\tau"] & \widetilde{Y} \arrow[d, "\pi_Y"] \\
X \arrow[r, "f"'] & f(X) \arrow[r] & Y
\end{tikzcd}$$

where $\widetilde{f}$ is surjective. Then the functoriality property holds for $\widetilde{f}$ by the argument above and for i_W by Theorem 4.14 (ii). This finishes the proof.

(ii) We can suppose that $x = [\mathcal{F}]$, $y = [\mathcal{G}]$ and that $\mathcal{F}$ and $\mathcal{G}$ admit locally free quotients $\mathcal{E}_1$, $\mathcal{E}_2$ of maximal rank. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the associated kernels. We can also suppose that $\mathrm{supp}(\mathcal{T}_1)$ lies in a simple normal crossing divisor. Then

$$\begin{aligned}
\mathrm{ch}([\mathcal{F}].[\mathcal{G}]) &= \overline{\mathrm{ch}}([\mathcal{E}_1].[\mathcal{E}_2]) + \mathrm{ch}([\mathcal{E}_1].[\mathcal{T}_2]) + \mathrm{ch}([\mathcal{E}_2].[\mathcal{T}_1]) + \mathrm{ch}([\mathcal{T}_1].[\mathcal{T}_2]) \\
&= \overline{\mathrm{ch}}(\mathcal{E}_1)\overline{\mathrm{ch}}(\mathcal{E}_2) + \overline{\mathrm{ch}}(\mathcal{E}_1)\,\mathrm{ch}(\mathcal{T}_2) + \overline{\mathrm{ch}}(\mathcal{E}_2)\,\mathrm{ch}(\mathcal{T}_1) + \mathrm{ch}([\mathcal{T}_1].[\mathcal{T}_2])
\end{aligned}$$

by (W_n), Proposition 4.2 (ii) and Proposition 3.7 (iii). By dévissage, we can suppose that $\mathcal{T}_1$ is a $\mathcal{O}_Z$-module, where Z is a smooth hypersurface of X. We write $[\mathcal{T}_1] = i_{Z!}u$ and $[\mathcal{T}_2] = v$. Then $[\mathcal{T}_1]\,.\,[\mathcal{T}_2] = i_{Z!}\big(u\,.\,i_Z^\dagger v\big)$. So

$$\begin{aligned}
\mathrm{ch}\big([\mathcal{T}_1].[\mathcal{T}_2]\big) &= i_{Z*}\Big(\mathrm{ch}(u\,.\,i_Z^\dagger v)\ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big) \\
&= i_{Z*}\Big(\mathrm{ch}(u)\ i_Z^*\,\mathrm{ch}(v)\ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big) && \text{by } (\mathrm{P}_{n-1}) \text{ and Proposition 4.9 (ii)} \\
&= i_{Z*}\Big(\mathrm{ch}(u)\ \mathrm{td}\big(N_{Z/X}\big)^{-1}\Big)\,\mathrm{ch}(v) && \text{by the projection formula} \\
&= \mathrm{ch}\big(i_{Z!}u\big)\,\mathrm{ch}(v) = \mathrm{ch}\big([\mathcal{T}_1]\big)\,\mathrm{ch}\big([\mathcal{T}_2]\big)
\end{aligned}$$

□

The proof of Theorem 4.1 is now concluded with the exception of property (RR_n).

6. The Grothendieck-Riemann-Roch theorem for projective morphisms

6.1. Proof of the GRR formula. We have already obtained the (GRR) formula for the immersion of a smooth divisor. We reduce now the general case to the divisor case by a blowup. This construction is classical (see [**Bo-Se**]).

Theorem 6.1. *Let Y be a smooth submanifold of X. Then, for all y in $K(Y)$, we have*

$$\mathrm{ch}\big(i_{Y!}y\big) = i_{Y*}\big(\mathrm{ch}(y)\,\mathrm{td}(N_{Z/X})^{-1}\big).$$

Proof. We blow up Y along X as shown below, where E is the exceptional divisor.

$$\begin{tikzcd}
E \arrow[r, "i_E"] \arrow[d, "q"'] & \widetilde{X} \arrow[d, "p"] \\
Y \arrow[r, "i_Y"'] & X
\end{tikzcd}$$

Let F be the excess conormal bundle of q defined by the exact sequence

$$0 \longrightarrow F \longrightarrow q^* N^*_{Y/X} \longrightarrow N^*_{E/\widetilde{X}} \longrightarrow 0.$$

If d is the codimension of Y in X, then $\operatorname{rank}(F) = d - 1$. Recall the following formulae:

(a) Excess formula in K-theory (Proposition 7.6 (ii)): for all y in $K(Y)$,

$$p^\dagger\, i_! y = i_{E!}\big(q^\dagger y \,.\, \lambda_{-1} F\big).$$

(b) Excess formula in Deligne cohomology (Proposition 3.5 (vi)): for each Deligne class β on Y,

$$p^*\, i_{Y*}\, \beta = i_{E*}\big(q^* \beta\ c_{d-1}(F^*)\big).$$

(c) If G is a vector bundle of rank r, then $\operatorname{ch}\big(\lambda_{-1}[G]\big) = c_r(G^*)\operatorname{td}(G^*)^{-1}$ ([**Bo-Se**, Lemme 18]).

We compute now

$$\begin{aligned}
p^* \operatorname{ch}\big(i_{Y!} y\big) &= \operatorname{ch}\big(p^\dagger\, i_{Y!} y\big) = \operatorname{ch}\big(i_{E!}(q^\dagger y \,.\, \lambda_{-1}[F])\big) && \text{by } (\mathrm{F}_n) \text{ and (a)}\\
&= i_{E*}\Big(\operatorname{ch}\big(q^\dagger y \,.\, \lambda_{-1}[F]\big)\operatorname{td}\big(N_{E/\widetilde{X}}\big)^{-1}\Big) && \text{by (GRR) for } i_E\\
&= i_{E*}\Big(q^* \operatorname{ch}(y)\operatorname{ch}\big(\lambda_{-1}[F]\big)\ q^* \operatorname{td}\big(N_{Y/X}\big)^{-1}\operatorname{td}\big(F^*\big)\Big) && \text{by } (\mathrm{F}_n),\ (\mathrm{P}_n) \text{ and}\\
& && \quad\text{Proposition 3.7 (i)}\\
&= i_{E*}\Big(q^*\big(\operatorname{ch}(y)\operatorname{td}\big(N_{Y/X}\big)^{-1}\big)\, c_{d-1}(F^*)\Big) && \text{by (c)}\\
&= p^*\, i_{Y*}\Big(\operatorname{ch}(y)\operatorname{td}\big(N_{Y/X}\big)^{-1}\Big) && \text{by (b).}
\end{aligned}$$

Thus $\operatorname{ch}\big(i_{Y!} y\big) = i_{Y*}\big(\operatorname{ch}(y)\operatorname{td}\big(N_{Y/X}\big)^{-1}\big)$. □

Now we can prove a more general Grothendieck-Riemann-Roch theorem:

THEOREM 6.2. *The GRR theorem holds in rational Deligne cohomology for projective morphisms between smooth complex compact manifolds.*

PROOF. Let $f\colon X \longrightarrow Y$ be a projective morphism. Then we can write f as the composition of an immersion $i\colon X \longrightarrow Y \times \mathbb{P}^N$ and the second projection $p\colon Y \times \mathbb{P}^N \longrightarrow Y$. By Theorem 6.1, GRR is true for i. To prove the GRR theorem for p, we use the following result:

PROPOSITION 6.3. [Bei] *For any complex manifold B and any $N \in \mathbb{N}^*$, the canonical map $K(B) \otimes_{\mathbb{Z}} K(\mathbb{P}^N) \longrightarrow K(B \times \mathbb{P}^N)$ is surjective.*

PROOF. We introduce some notations:

- $\Delta \subseteq \mathbb{P}^N \times \mathbb{P}^N$ is the diagonal,
- $i\colon \mathbb{P}^N \times B \longrightarrow \mathbb{P}^N \times \mathbb{P}^N \times B$ is the diagonal injection over B,
- $\pi\colon \mathbb{P}^N \times B \longrightarrow B$ is the second projection,
- $p_1,\, p_2 : \mathbb{P}^N \times \mathbb{P}^N \times B \longrightarrow \mathbb{P}^N \times B$ are the projections defined by $p_1(x, y, b) = (x, b)$ and $p_2(x, y, b) = (y, b)$.

If $\lambda \in K(B \times \mathbb{P}^N)$, then

$$\lambda = (p_2 \circ i)_!\, \lambda = p_{2\,!}\, i_!(p_1 \circ i)^\dagger \lambda = p_{2\,!}(i_!\, i^\dagger\, p_1^\dagger \lambda) = p_{2\,!}(p_1^\dagger \lambda \,.\, \mathcal{O}_{\Delta \times B}).$$

On $\mathbb{P}^N \times \mathbb{P}^N$, we have a canonical resolution of $\mathcal{O}_\Delta$ by the Koszul exact sequence:

$$0 \longrightarrow \Omega^N(N) \boxtimes_{\mathcal{O}} \mathcal{O}(-N) \longrightarrow \cdots \longrightarrow \Omega^1(1) \boxtimes_{\mathcal{O}} \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0.$$

We can pullback this exact sequence on $\mathbb{P}^N \times \mathbb{P}^N \times B$. This shows that in $K(\mathbb{P}^N \times \mathbb{P}^N \times B)$, $\mathcal{O}_{\Delta\times B} = \sum_{i=0}^{N}(-1)^i p_1^\dagger\big(\Omega^i(i) \boxtimes_{\mathcal{O}} \mathcal{O}_B\big) \otimes_{\mathcal{O}} p_2^\dagger\big(\mathcal{O}(-i) \boxtimes_{\mathcal{O}} \mathcal{O}_B\big)$. We can now consider the diagram

$$\begin{tikzcd}
\mathbb{P}^N \times \mathbb{P}^N \times B \arrow[r, "p_1"] \arrow[d, "p_2"'] & \mathbb{P}^N \times B \arrow[d, "\pi"] \\
\mathbb{P}^N \times B \arrow[r, "\pi"'] & B
\end{tikzcd}$$

Then

$$\begin{aligned}
p_{2\,!}\big(p_1^\dagger\lambda\,.\,\mathcal{O}_{\Delta\times B}\big) &= \sum_{i=0}^{N}(-1)^i\, p_{2\,!}\Big(p_1^\dagger\big(\lambda\,.\,[\Omega^i(i) \boxtimes_{\mathcal{O}} \mathcal{O}_B]\big)\,.\,p_2^\dagger\big(\mathcal{O}(-i) \boxtimes_{\mathcal{O}} \mathcal{O}_B\big)\Big) \\
&= \sum_{i=0}^{N}(-1)^i\,[\mathcal{O}(-i) \boxtimes_{\mathcal{O}} \mathcal{O}_B]\,.\,p_{2\,!}\,p_1^\dagger\big(\lambda\,.\,[\Omega^i(i) \boxtimes_{\mathcal{O}} \mathcal{O}_B]\big) \\
&= \sum_{i=0}^{N}(-1)^i[\mathcal{O}(-i) \boxtimes_{\mathcal{O}} \mathcal{O}_B]\,.\,\pi^\dagger\,\pi_!\big(\lambda\,.\,[\Omega^i(i) \boxtimes_{\mathcal{O}} \mathcal{O}_B]\big) \\
&= \sum_{i=0}^{N}(-1)^i\mathcal{O}(-i) \boxtimes_{\mathcal{O}}\ \pi_!\big(\lambda\,.\,[\Omega^i(i) \boxtimes_{\mathcal{O}} \mathcal{O}_B]\big)
\end{aligned}$$

which is in the image of the map $K(B) \otimes_{\mathbb{Z}} K(\mathbb{P}^N) \longrightarrow K(B \times \mathbb{P}^N)$. □

Therefore, it is enough to prove GRR for p with elements of the form $y\,.\,w$, where y belongs to $K(Y)$ and w belongs to $K(\mathbb{P}^N)$. By the product formula for the Chern character, we are led to the Hirzebruch-Riemann-Roch formula for $\mathbb{P}^N$, which is well known. □

REMARK 6.4. This proof shows that the GRR theorem holds for $p\colon Y \times Z \longrightarrow Y$ if Y is an arbitrary complex compact manifold and Z is a complex compact manifold such that the class of structure sheaf of the diagonal of $Z \times Z$ in $K(Z \times Z)$ lies in the image of the natural morphism $K(Z) \otimes_{\mathbb{Z}} K(Z) \longrightarrow K(Z \times Z)$. This is indeed the case for $Z = \mathbb{P}^N$.

6.2. Compatibility of Chern classes and the GRR formula. We will show that the GRR formula for immersions combined with some basic properties can be sufficient to characterize completely a theory of Chern classes. This will give various compatibility theorems.

We assume to be given for each smooth complex compact manifold X a graded commutative cohomology ring $A(X) = \oplus_{i=0}^{\dim X} A^i(X)$ which is an algebra over $\mathbb{Q}$, $\mathbb{Q} \subset A^0(X)$, with the following properties:

(α) For each holomorphic map $f : X \longrightarrow Y$, there exists a pull-back morphism $f^* : A(Y) \longrightarrow A(X)$ which is functorial and compatible with the products and the gradings.

(β) If σ is the blowup of a smooth complex compact manifold along a smooth submanifold, then σ^* is injective.

(γ) If E is a holomorphic vector bundle on X and $\pi\colon \mathbb{P}(E) \longrightarrow X$ is the projection of the associated projective bundle, then π^* is injective.

(δ) If X is a smooth complex compact manifold and Y is a smooth submanifold of codimension d, then there is a Gysin morphism $i_*\colon A^*(Y) \longrightarrow A^{*+d}(X)$.

The main examples are:

$$A^i(X) = H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i)),\ \mathbb{H}^{2i}(X, \Omega_X^{\bullet \geqslant i}),\ H^i(X, \Omega_X^i),\ F^i H^{2i}(X, \mathbb{C}),\ H^{2i}(X, \mathbb{Q}) \text{ and } H^{2i}(X, \mathbb{C}).$$

Then we have the following theorem:

THEOREM 6.5. *Suppose that we have two theories of Chern classes c_i and c'_i for coherent sheaves on any smooth complex compact manifold X with values in $A^i(X)$ such that $c_0 = c'_0 = 1$ and*

(i) *The Whitney formula holds for the total classes c and c'.*

(ii) *The functoriality formula holds for c and c'.*

(iii) *If L is a holomorphic line bundle, then*

$$c(L) = 1 + c_1(L) = c'(L) = 1 + c'_1(L).$$

(iv) *In both theories, the GRR theorem holds for immersions.*

Then for every coherent sheaf $\mathcal{F}$, $c(\mathcal{F})$ and $c'(\mathcal{F})$ are equal.

REMARK 6.6.

1. The same conclusion holds for cohomology algebras over $\mathbb{Z}$ if we assume GRR *without denominators.*
2. If X is projective, (i) and (iii) are sufficient to imply the equality of c and c' because of the existence of global locally free resolutions.
3. In (iv), the Todd classes of the normal bundle are defined in $A(X)$ since $A(X)$ is a $\mathbb{Q}$-algebra.

PROOF. We start by proving that for any holomorphic vector bundle E, $c(E)$ and $c'(E)$ are equal. Actually, this proof is a variant of the splitting principle. We argue by induction on the rank of E. Let $\pi\colon \mathbb{P}(E) \longrightarrow X$ be the projective bundle of E. Then we have the exact sequence

$$0 \longrightarrow \mathcal{O}_E(-1) \longrightarrow \pi^* E \longrightarrow F \longrightarrow 0$$

on $\mathbb{P}(E)$, where F is a holomorphic vector bundle on $\mathbb{P}(E)$ whose rank is the rank of E minus one. By induction, $c(F) = c'(F)$ and by (iii), $c(\mathcal{O}_E(-1)) = c'(\mathcal{O}_E(-1))$. By (i), $c(\pi^* E) = c'(\pi^* E)$ and by (ii), $\pi^*[c(E) - c'(E)] = 0$. By (γ), $c(E) = c'(E)$.

We can now prove Theorem 6.5. As usual, we deal with exponential Chern classes. The proof proceeds by induction on the dimension of the base manifold X.

Let $\mathcal{F}$ be a coherent sheaf on X. By Theorem 4.10 there exists a bimeromorphic morphism $\sigma\colon \widetilde{X} \longrightarrow X$ which is a finite composition of blowups with smooth centers and a locally free sheaf $\mathcal{E}$ on $\widetilde{X}$ which is a quotient of maximal rank of $\sigma^*\mathcal{F}$. Furthermore, by Hironaka's theorem, we can suppose that the exceptional locus of σ and the kernel of the morphism $\sigma^*\mathcal{F} \longrightarrow \mathcal{E}$ are both contained in a simple normal crossing divisor D of $\widetilde{X}$. Thus

$$\sigma^\dagger[\mathcal{F}] = \sum_{i=0}^{n} (-1)^i \big[\mathrm{Tor}_i(\mathcal{F}, \sigma)\big] = [\mathcal{E}] + \sum_{i=1}^{n} (-1)^i \big[\mathrm{Tor}_i(\mathcal{F}, \sigma)\big]$$

and then $\sigma^{\dagger}[\mathcal{F}]$ belongs to $[\mathcal{E}] + K_D(\widetilde{X})$. Now there is a surjective morphism

$$\bigoplus_{i=1}^{N} K_{D_i}(\widetilde{X}) \longrightarrow K_D(\widetilde{X}) \ .$$

Moreover, by Proposition 7.2, $K(D_i) \simeq K_{D_i}(\widetilde{X})$. Remark that $\mathrm{td}\big(N_{D_i/\widetilde{X}}\big) = \mathrm{td}'\big(N_{D_i/\widetilde{X}}\big)$. By the GRR formulae (iv) and the induction hypothesis, ch and ch′ are equal on each $K_{D_i}(\widetilde{X})$. By the first part of the proof, $\mathrm{ch}(\mathcal{E}) = \mathrm{ch}'(\mathcal{E})$. Thus $\mathrm{ch}\big(\sigma^{\dagger}[\mathcal{F}]\big) = \mathrm{ch}'\big(\sigma^{\dagger}[\mathcal{F}]\big)$. By (ii), $\sigma^*\big[\mathrm{ch}(\mathcal{F}) - \mathrm{ch}'(\mathcal{F})\big] = 0$. Since σ^* is injective by (β), $\mathrm{ch}(\mathcal{F}) = \mathrm{ch}'(\mathcal{F})$. □

Corollary 6.7. *Let $\mathcal{F}$ be a coherent analytic sheaf on X. Then:*

(i) *The classes $c_i(\mathcal{F})$ in $H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i))$ and $c_i(\mathcal{F})^{top}$ in $H^{2i}(X, \mathbb{Z})$ have the same image in $H^{2i}(X, \mathbb{Q})$.*

(ii) *The image of $c_i(\mathcal{F})$ via the natural morphism from $H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i))$ to $H^i(X, \Omega^i_X)$ is the i-th Atiyah Chern class of $\mathcal{F}$*

Proof. If L is a holomorphic line bundle, then (i) and (ii) hold for L. Indeed, using the isomorphism between $\mathrm{Pic}(X)$ and $H^2_{\mathrm{Del}}(X, \mathbb{Z}(1))$, the topological and Atiyah first Chern classes are obtained by the two morphisms of complexes

$$\begin{array}{ccc} \mathbb{Z}_X \xrightarrow{2i\pi} \mathcal{O}_X & \qquad \text{and} \qquad & \mathbb{Z}_X \xrightarrow{2i\pi} \mathcal{O}_X \\ \downarrow \qquad\quad & & \qquad \downarrow{\scriptstyle d/2i\pi} \\ \mathbb{Z}_X \qquad\quad & & \qquad \Omega^1_X \end{array}$$

On the other hand, GRR for immersions holds for topological Chern classes by [**At-Hi**] and for Atiyah Chern classes by [**OB-To-To 3**]. □

Remark 6.8. If X is a Kähler complex manifold, the Green Chern classes are the same as the Atiyah Chern classes and the complex topological Chern classes. If X is non Kähler, GRR does not seem to be known for the Green Chern classes, except for a constant morphism (see [**To-To**]). If this were true for immersions, it would imply the compatibility of $c_i(\mathcal{F})$ and $c_i(\mathcal{F})^{\mathrm{Gr}}$, via the map from $H^{2i}_{\mathrm{Del}}(X, \mathbb{Q}(i))$ to $\mathbb{H}^{2i}(X, \Omega_X^{\bullet \geqslant i})$. On the other hand, if this compatibility holds, it implies GRR for immersions for the Green Chern classes.

7. Appendix. Analytic K-theory with support

Our aim in this appendix is to answer the following question: if $f\colon X \longrightarrow Y$ is a morphism, $\mathcal{F}$ a torsion sheaf on Y with support W, and $Z = f^{-1}(W)$, can we express the sheaves $\mathrm{Tor}_i(\mathcal{F}, f)$ in terms of the sheaves $\mathrm{Tor}_i(\mathcal{F}_{|W}, f_{|Z})$? Recall that $\mathrm{Tor}_i(\mathcal{F}, f) = \mathrm{Tor}^i_{f^{-1}\mathcal{O}_Y}\big(f^{-1}\mathcal{F}, \mathcal{O}_X\big)$ is the i-th left derived functor of the functor $F\colon \mathrm{Mod}(\mathcal{O}_Y) \longrightarrow \mathrm{Mod}(\mathcal{O}_X)$ given by $F(\mathcal{F}) = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. These derived functors exist thanks to the existence of global flat resolutions for any sheaf of rings (see [**Ka-Sc**] page 110). Furthermore, if $\mathcal{F}$ is coherent on Y, the sheaves $\mathrm{Tor}_i(\mathcal{F}, f)$ are coherent on X. This result is the consequence of the existence of locally free resolutions of finite rank of $\mathcal{F}$.

7.1. Definition of the analytic K-theory with support. Let X be a smooth compact complex manifold and Z be an analytic subset of X. We will denote by $\mathrm{coh}_Z(X)$ the abelian category of coherent sheaves on X with support in Z. Then, X being compact,

$$\mathrm{coh}_Z(X) = \big\{\mathcal{F},\ \mathcal{F} \in \mathrm{coh}(X) \text{ such that } \mathcal{I}_Z^n \mathcal{F} = 0 \text{ for } n \gg 0\big\}\cdot$$

Definition 7.1. We define $K_Z(X)$ as the quotient of $\mathbb{Z}\big[\mathrm{coh}_Z(X)\big]$ by the relations: $\mathcal{F}+\mathcal{H}=\mathcal{G}$ if there exists an exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ with $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ elements of $\mathrm{coh}_Z(X)$. In other words, $K_Z(X)$ is the Grothendieck group of $\mathrm{coh}_Z(X)$. The image of an element $\mathcal{F}$ of $\mathrm{coh}_Z(X)$ in $K_Z(X)$ will be denoted by $[\mathcal{F}]$.

We first prove

Proposition 7.2. *The map* $i_{Z*}\colon K(Z) \longrightarrow K_Z(X)$ *is a group isomorphism.*

Proof. We consider the inclusion $\mathrm{coh}(Z) \subseteq \mathrm{coh}_Z(X)$. If $\mathcal{F}$ is in $\mathrm{coh}_Z(X)$, we can filter $\mathcal{F}$ by the formula $F^i\mathcal{F} = \mathcal{I}_Z^i\mathcal{F}$. This is a finite filtration and the associated quotients $F^i\mathcal{F}/_{F^{i+1}\mathcal{F}}$ are in $\mathrm{coh}(Z)$. Therefore, the proposition is a consequence of the dévissage theorem for the Grothendieck group, which is in fact valid for all K-theory groups (see [**Qui**, § 5, Theorem 4]). For the sake of completeness, we give here the details of the proof in this particular case.

If $\mathcal{F}$ is an element of $\mathrm{coh}_Z(X)$, we define $l(\mathcal{F})$ as the smallest $n \geq 1$ for which $\mathcal{I}_Z^n\mathcal{F} = 0$. We define $\phi\colon \mathbb{Z}\big[\mathrm{coh}_Z(X)\big] \longrightarrow K(Z)$ by

$$\phi(\mathcal{F}) = \sum_{i\geq 0} \mathrm{Gr}^i(\mathcal{F})_{|Z}$$

where $\mathcal{F}$ is filtered by $\big(\mathcal{I}_Z^i\mathcal{F}\big)_{i\geq 0}$ and $\mathrm{Gr}^i(\mathcal{F}) = \mathcal{I}_Z^i\mathcal{F}/_{\mathcal{I}_Z^{i+1}\mathcal{F}}$.

Let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ be an exact sequence of elements of $\mathrm{coh}_Z(X)$. We want to show that $\phi(\mathcal{F}) + \phi(\mathcal{H}) = \phi(\mathcal{G})$.

Step 1. $l(\mathcal{F}) = l(\mathcal{G}) = l(\mathcal{H}) = 1$. This means that $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ are sheaves of $\mathcal{O}_Z$-modules. Then $0 \longrightarrow i_Z^*\mathcal{F} \longrightarrow i_Z^*\mathcal{G} \longrightarrow i_Z^*\mathcal{H} \longrightarrow 0$ is exact, and $\phi(\mathcal{F}) + \phi(\mathcal{H}) = \phi(\mathcal{G})$.

Step 2. $l(\mathcal{H}) = 1$. We proceed by induction on $l(\mathcal{F}) + l(\mathcal{G})$. We have an exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{G}_{|Z} \longrightarrow \mathcal{H} \longrightarrow 0\,.$$

Since $l(\mathcal{N}) = l(\mathcal{G}_{|Z}) = l(\mathcal{H}) = 1$, $\phi(\mathcal{G}_{|Z}) = \phi(\mathcal{N}) + \phi(\mathcal{H})$. Besides, we have an exact sequence

$$0 \longrightarrow \mathcal{I}_Z\mathcal{G} \longrightarrow \mathcal{F} \longrightarrow \mathcal{N} \longrightarrow 0,$$

and by induction $\phi(\mathcal{F}) = \phi(\mathcal{I}_Z\mathcal{G}) + \phi(\mathcal{N})$. We finally obtain $\phi(\mathcal{G}) = \phi(\mathcal{F}) + \phi(\mathcal{H})$ by remarking that $\phi(\mathcal{G}) = \phi(\mathcal{G}_{|Z}) + \phi(\mathcal{I}_Z\mathcal{G})$.

Step 3. General case. We proceed by induction on $l(\mathcal{H})$. We have an exact sequence

$$0 \longrightarrow \mathcal{I}_Z\mathcal{G} \cap \mathcal{F} \longrightarrow \mathcal{I}_Z\mathcal{G} \longrightarrow \mathcal{I}_Z\mathcal{H} \longrightarrow 0\,.$$

By induction $\phi(\mathcal{I}_Z\mathcal{G}) = \phi(\mathcal{I}_Z\mathcal{G} \cap \mathcal{F}) + \phi(\mathcal{I}_Z\mathcal{H})$. We have an exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H}/_{\mathcal{I}_Z\mathcal{H}} \longrightarrow 0\,.$$

By Step 2, $\phi(\mathcal{G}) = \phi(\mathcal{N}) + \phi(\mathcal{H}/_{\mathcal{I}_Z\mathcal{H}})$. Remark that we have another exact sequence, namely

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{N} \longrightarrow \mathcal{I}_Z\mathcal{G}/_{\mathcal{I}_Z\mathcal{G} \cap \mathcal{F}} \longrightarrow 0\,.$$

Let $N = l(\mathcal{H})$. Since $\mathcal{I}_Z^N\mathcal{G} \subseteq \mathcal{F}$, we have $l\big(\mathcal{I}_Z\mathcal{G}/_{\mathcal{I}_Z\mathcal{G}\cap\mathcal{F}}\big) \leq N-1 = l(\mathcal{H})-1$. By induction $\phi(\mathcal{N}) = \phi(\mathcal{F}) + \phi\big(\mathcal{I}_Z\mathcal{G}/_{\mathcal{I}_Z\mathcal{G}\cap\mathcal{F}}\big)$ and $\phi(\mathcal{I}_Z\mathcal{G}) = \phi(\mathcal{I}_Z\mathcal{G}\cap\mathcal{F}) + \phi\big(\mathcal{I}_Z\mathcal{G}/_{\mathcal{I}_Z\mathcal{G}\cap\mathcal{F}}\big)$. Collecting all these relations, we get $\phi(\mathcal{G}) = \phi(\mathcal{F}) + \phi(\mathcal{H})$. $\square$

7.2. Product on the K-theory with support. Let Z be a *smooth* submanifold of X. Recall that $K(Z)$ is a commutative unitary ring, the product being defined by

$$[\mathcal{F}]\,.\,[\mathcal{G}] = \sum_i (-1)^i \big[\mathrm{Tor}^i_{\mathcal{O}_Z}(\mathcal{F},\mathcal{G})\big].$$

The unit element is the class of $\mathcal{O}_Z$. For any x in $K_Z(X)$, by Proposition 7.2, there exists a unique $\overline{x}$ in $K(Z)$ such that $i_{Z*}\overline{x} = x$.

Definition 7.3. We define a product $K(Z)\otimes_{\mathbb{Z}} K_Z(X) \xrightarrow[\centerdot_Z]{} K_Z(X)$ by $x\centerdot_Z y = i_{Z*}(x.\overline{y})$.

In other words, the following diagram is commutative:

$$\begin{tikzcd}
K(Z)\otimes_{\mathbb{Z}} K_Z(X) \arrow[r, "\centerdot_Z"'] \arrow[d, "\simeq"] & K_Z(X) \arrow[d, "\simeq"] \\
K(Z)\otimes_{\mathbb{Z}} K(Z) \arrow[r, "."'] & K(Z)
\end{tikzcd}$$

Remark 7.4. Definition 7.3 has some obvious consequences:

1. For any x in $K(Z)$, $i_{Z*}x = x \centerdot_Z [i_{Z*}\mathcal{O}_Z]$.
2. More generally, $i_{Z*}(x.y) = x \centerdot_Z i_{Z*}y$.
3. The product $\centerdot_Z$ endows $K_Z(X)$ with the structure of a $K(Z)$-module.

7.3. Functoriality. Let $f\colon X \longrightarrow Y$ be a holomorphic map, W a smooth submanifold of Y and $Z = f^{-1}(W)$. We can define a morphism $f^{\dagger}\colon K_W(Y) \longrightarrow K_Z(X)$ by the usual formula $f^{\dagger}[\mathcal{K}] = \sum_{i\geq 0}(-1)^i\big[\mathrm{Tor}_i(\mathcal{K}, f)\big]$. Let $\overline{f}$ be the restriction of f to Z, as shown in the following diagram:

$$\begin{tikzcd}
Z \arrow[r, "i_Z"] \arrow[d, "\overline{f}"'] & X \arrow[d, "f"] \\
W \arrow[r, "i_W"'] & Y
\end{tikzcd}$$

Then we have the following proposition:

Proposition 7.5. *Suppose that Z is smooth. For all x in $K(W)$ and for all y in $K_W(Y)$, we have*

$$f^{\dagger}\big(x \centerdot_W y\big) = \overline{f}^{\,\dagger}x \centerdot_Z f^{\dagger}y.$$

Proof. We consider the functor $\mathcal{F} \longrightarrow f^*i_{W*}\mathcal{F}$ from $\mathrm{Mod}(\mathcal{O}_W)$ to $\mathrm{Mod}(\mathcal{O}_X)$.

$$\begin{aligned}
f^*(i_{W*}\mathcal{F}) &= f^{-1}i_{W*}\mathcal{F}\otimes_{f^{-1}\mathcal{O}_Y}\mathcal{O}_X = i_{Z*}\overline{f}^{-1}\mathcal{F}\otimes_{f^{-1}\mathcal{O}_Y}\mathcal{O}_X \\
&= i_{Z*}\overline{f}^{-1}\mathcal{F}\otimes_{f^{-1}\mathcal{O}_Y/f^{-1}\mathcal{I}_W}(\mathcal{O}_X/f^{-1}\mathcal{I}_W.\mathcal{O}_X) \\
&= i_{Z*}\overline{f}^{-1}\mathcal{F}\otimes_{i_{Z*}\overline{f}^{-1}\mathcal{O}_W}(\mathcal{O}_X/f^{-1}\mathcal{I}_W.\mathcal{O}_X).
\end{aligned}$$

Thus $\mathrm{Tor}^i_{f^{-1}\mathcal{O}_Y}\big(f^{-1}i_{W*}\mathcal{F},\mathcal{O}_X\big) = \mathrm{Tor}^i_{i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W}\big(i_{Z*}\overline{f}^{\,-1}\mathcal{F},\mathcal{U}\big)$ for $i\geq 1$. Now we can filter the sheaf $\mathcal{U}$ as in the proof of Proposition 7.2 by $F^i\mathcal{U} = \mathcal{I}_Z^i\mathcal{U}$. We have exact sequences

$$0 \longrightarrow F^{l+1}\mathcal{U} \longrightarrow F^l\mathcal{U} \longrightarrow \mathrm{Gr}^l\,\mathcal{U} \longrightarrow 0$$

Remark that $F^l\mathcal{U}$ is a sheaf of $i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W$-modules. Then we obtain a long exact sequence of coherent sheaves supported in Z:

$$\cdots \longrightarrow \operatorname{Tor}^k_{i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W}\big(i_{Z*}\overline{f}^{\,-1}\mathcal{F}, F^{l+1}\mathcal{U}\big) \longrightarrow \operatorname{Tor}^k_{i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W}\big(i_{Z*}\overline{f}^{\,-1}\mathcal{F}, F^{l}\mathcal{U}\big)$$
$$\longrightarrow \operatorname{Tor}^k_{i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W}\big(i_{Z*}\overline{f}^{\,-1}\mathcal{F}, \operatorname{Gr}^l\mathcal{U}\big) \longrightarrow \operatorname{Tor}^{k+1}_{i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W}\big(i_{Z*}\overline{f}^{\,-1}\mathcal{F}, F^{l+1}\mathcal{U}\big) \longrightarrow \cdots$$

Thus, in $K_Z(X)$, we have $f^{\dagger}[i_{Z*}\mathcal{F}] = \displaystyle\sum_{i,l\geq 0}(-1)^i\big[\operatorname{Tor}^i_{i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W}\big(i_{Z*}\overline{f}^{\,-1}\mathcal{F}, \operatorname{Gr}^l\mathcal{U}\big)\big]$. The graded pieces $\operatorname{Gr}^l\mathcal{U}$ are sheaves of $i_{Z*}\mathcal{O}_Z$-modules. If $\mathcal{K}$ is any coherent sheaf of $\mathcal{O}_Z$-modules on Z, then

$$i_{Z*}\overline{f}^{\,-1}\mathcal{F} \otimes_{i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W} i_{Z*}\mathcal{K} = i_{Z*}\big(\overline{f}^{\,*}\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{K}\big).$$

Therefore we have a spectral sequence such that

$$E_2^{p,q} = i_{Z*}\operatorname{Tor}^p_{\mathcal{O}_Z}\Big(\operatorname{Tor}^q_{\overline{f}^{\,-1}\mathcal{O}_W}\big(\overline{f}^{\,-1}\mathcal{F}, \mathcal{O}_Z\big), \mathcal{K}\Big)$$
$$E_\infty^{p,q} = \operatorname{Gr}^p\operatorname{Tor}^{p+q}_{i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W}\big(i_{Z*}\overline{f}^{\,-1}\mathcal{F}, i_{Z*}\mathcal{K}\big).$$

This gives in $K_Z(X)$ the equality

$$\sum_{i\geq 0}(-1)^i\big[\operatorname{Tor}^i_{i_{Z*}\overline{f}^{\,-1}\mathcal{O}_W}\big(i_{Z*}\overline{f}^{\,-1}\mathcal{F}, i_{Z*}\mathcal{K}\big)\big] \simeq i_{Z*}\big(\overline{f}^{\,\dagger}[\mathcal{F}]\,.\,[\mathcal{K}]\big) = \overline{f}^{\,\dagger}[\mathcal{F}]_{\bullet_Z}[i_{Z*}\mathcal{K}].$$

Applying this to $\mathcal{K} = \operatorname{Gr}^l\mathcal{U}$, we obtain

$$f^{\dagger}[i_{Z*}\mathcal{F}] = \sum_{l\geq 0}\overline{f}^{\,\dagger}[\mathcal{F}]_{\bullet_Z}[\operatorname{Gr}^l\mathcal{U}] = \overline{f}^{\,\dagger}[\mathcal{F}]_{\bullet_Z}[\mathcal{U}].$$

We choose $\mathcal{F} = \mathcal{O}_W$. Then $\mathcal{U} = f^{\dagger}[i_{W*}\mathcal{O}_W]$ so that $f^{\dagger}[i_{W*}\mathcal{F}] = \overline{f}^{\,\dagger}[\mathcal{F}]_{\bullet_Z} f^{\dagger}[i_{W*}\mathcal{O}_W]$. Thus $\forall x \in K(W)$, $\forall y \in K_W(Y)$, we have

$$\begin{aligned} f^{\dagger}(x_{\bullet_W} y) &= f^{\dagger}(i_{W*}(x.\overline{y})) = \overline{f}^{\,\dagger}(x.\overline{y})_{\bullet_Z} f^{\dagger}[i_{W*}\mathcal{O}_W] \\ &= (\overline{f}^{\,\dagger}x.\overline{f}^{\,\dagger}\overline{y})_{\bullet_Z} f^{\dagger}[i_{W*}\mathcal{O}_W] = \overline{f}^{\,\dagger}x_{\bullet_Z}(\overline{f}^{\,\dagger}\overline{y}_{\bullet_Z} f^{\dagger}[i_{W*}\mathcal{O}_W]) \\ &= \overline{f}^{\,\dagger}x_{\bullet_Z} f^{\dagger}(i_{W*}\overline{y}) = \overline{f}^{\,\dagger}x_{\bullet_Z} f^{\dagger}y. \end{aligned}$$

□

Recall that if E is a holomorphic vector bundle on X, $\lambda_{-1}[E]$ is the element of $K(X)$ defined by

$$\lambda_{-1}[E] = 1 - [E] + \big[\textstyle\bigwedge^2 E\big] - \big[\textstyle\bigwedge^3 E\big] + \cdots$$

Proposition 7.6. (In the algebraic case, see [**Bo-Se**, Proposition 12 and Lemme 19 c.])

(i) *If Z is a smooth submanifold of X, then for all x in $K(Z)$, $i_Z^{\dagger}i_{Z*}x = x\,.\,\lambda_{-1}[N^*_{Z/X}]$.*

(ii) *Consider the blowup $\widetilde{X}$ of X along a smooth submanifold Y, and let E be the exceptional divisor as shown in the following diagram:*

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "q"'] & \widetilde{X} \arrow[d, "p"] \\ Y \arrow[r, "i"'] & X \end{tikzcd}$$

Let F be the excess conormal bundle on E defined by the exact sequence

$$0 \longrightarrow F \longrightarrow q^*N^*_{Y/X} \longrightarrow N^*_{E/\widetilde{X}} \longrightarrow 0.$$

Then for all x in $K(Y)$, $p^\dagger i_* x = j_*\big(q^\dagger x \,.\, \lambda_{-1}[F]\big)$.

The assertion (ii) of Proposition 7.6 is the *excess formula* in K-theory.

Proof. (i) We write $i_Z^\dagger i_{Z*}x = i_Z^\dagger\big(x \,\boldsymbol{.}_Z\, [i_{Z*}\mathcal{O}_Z]\big) = x \,.\, i_Z^\dagger [i_{Z*}\mathcal{O}_Z]$. Now, by [**Bo-Se**, Proposition 12], $\mathrm{Tor}_i(i_{Z*}\mathcal{O}_Z, i_Z) = \bigwedge^i N^*_{Z/X}$ and we are done.

(ii) The equality $p^\dagger\big[i_*\mathcal{O}_Y\big] = j_!(\lambda_{-1}[F])$ is proved in [**Bo-Se**, Lemme 19.c]. Thus

$$p^\dagger i_* x = p^\dagger\big(x \,\boldsymbol{.}_Y\, [i^*\mathcal{O}_Y]\big) = q^\dagger x \,\boldsymbol{.}_{\widetilde{Y}}\, p^\dagger [i_*\mathcal{O}_Y] = q^\dagger x \,\boldsymbol{.}_{\widetilde{Y}}\, j_*\big(\lambda_{-1}[F]\big) = j_*\big(q^\dagger x \,.\, \lambda_{-1}[F]\big).$$

□

We will now need a statement similar to Proposition 7.5, with the hypothesis that Z is a divisor with simple normal crossing.

Let $f : X \longrightarrow Y$ be a surjective holomorphic map, D be a smooth hypersurface of Y, and $\widetilde{D} = f^*(D)$. We suppose that $\widetilde{D}$ is a divisor of X with simple normal crossing. Let $\widetilde{D}_k$, $1 \leq k \leq N$, be the branches of $\widetilde{D}^{\mathrm{red}}$. We denote by $\overline{f}_k$ the restricted map $f : \widetilde{D}_k \longrightarrow D$. We can generalize Proposition 7.5 as follows:

Proposition 7.7. *For all elements u_k in $K_{\widetilde{D}_k}(X)$ such that $\big[\mathcal{O}_{\widetilde{D}}\big] = u_1 + \cdots + u_N$ in $K_{\widetilde{D}}(X)$ and for all y in $K(D)$,*

$$f^\dagger\big(i_{D*}\, y\big) = \sum_{k=1}^N \overline{f}_k^{\,\dagger}(y) \,\boldsymbol{.}_{\widetilde{D}_k}\, u_k$$

Proof. Let $\overline{f} = f_{|\widetilde{D}} : \widetilde{D} \longrightarrow D$ and let $\mathcal{F}$ be a coherent sheaf on D. By the proof of Proposition 7.5,

$$f^\dagger[i_{\widetilde{D}*}\mathcal{F}] = \sum_{i,l\geq 0} (-1)^i \big[\mathrm{Tor}^i_{i_{\widetilde{D}*}\overline{f}^{\,-1}\mathcal{O}_D}\big(i_{\widetilde{D}*}\overline{f}^{\,-1}\mathcal{F}, \mathrm{Gr}^l\,\mathcal{U}\big)\big]$$

in $K_{\widetilde{D}}(X)$, where $\mathcal{U} = \mathcal{O}_X/_{f^{\,-1}\mathcal{I}_D\,.\,\mathcal{O}_X}$. Here $f^{\,-1}\mathcal{I}_D\,.\,\mathcal{O}_X = \mathcal{I}_{\widetilde{D}}$, so that $\mathcal{U} = \mathcal{O}_{\widetilde{D}}$. Let us consider the map $\Gamma : \mathbb{Z}[\,\mathrm{coh}(\widetilde{D})\,] \longrightarrow K_{\widetilde{D}}(X)$ defined by

$$\Gamma(\mathcal{K}) = \sum_{i\geq 0} (-1)^i \big[\mathrm{Tor}^i_{i_{\widetilde{D}*}\overline{f}^{\,-1}\mathcal{O}_D}\big(i_{\widetilde{D}*}\overline{f}^{\,-1}\mathcal{F}, i_{\widetilde{D}*}\mathcal{K}\big)\big].$$

By the long Tor exact sequence, Γ induces a map $K(\widetilde{D}) \longrightarrow K_{\widetilde{D}}(X)$. We will still denote it by Γ. By the proof of Proposition 7.2, the inverse ϕ of the map $i_{\widetilde{D}*} : K(\widetilde{D}) \longrightarrow K_{\widetilde{D}}(X)$ is given by $\phi([\mathcal{G}]) = \sum_{l\geq 0}\big[\mathrm{Gr}^l\,\mathcal{G}_{|Z}\big]$. Thus $\sum_{l\geq 0}\big[\mathrm{Gr}^l\,\mathcal{U}\big] = i_{Z*}\phi([\mathcal{U}]) = i_{Z*}\phi([\mathcal{O}_{\widetilde{D}}])$ and $f^\dagger\big[i_{\widetilde{D}*}\mathcal{F}\big] = \Gamma\big(\phi([\mathcal{O}_{\widetilde{D}}])\big)$.

Now $\big[\mathcal{O}_{\widetilde{D}}\big] = u_1 + \cdots + u_N$ in $K_{\widetilde{D}_i}(X)$. Let $\overline{u}_i \in K(\widetilde{D}_i)$ be such that $i_{\widetilde{D}_i*}\overline{u}_i = u_i$. Then

$$f^\dagger\big[i_{\widetilde{D}*}\mathcal{F}\big] = \Gamma\Big[\phi\big(i_{\widetilde{D}_i/\widetilde{D}*}\overline{u}_1\big) + \cdots + \phi\big(i_{\widetilde{D}_i/\widetilde{D}*}\overline{u}_N\big)\Big].$$

If $\mathcal{L}$ is a coherent sheaf on $\widetilde{D}_i$, then

$$i_{\widetilde{D}_i*}\overline{f}^{\,-1}\mathcal{F} \otimes_{i_{\widetilde{D}*}\overline{f}^{\,-1}\mathcal{O}_D} i_{\widetilde{D}_i*}\mathcal{L} = i_{\widetilde{D}_i*}\big(\overline{f}_i^{\,*}\mathcal{F} \otimes_{\mathcal{O}_{\widetilde{D}_i}} \mathcal{L}\big).$$

Then we have a spectral sequence which satisfies

$$E_2^{p,q} = i_{\widetilde{D}_i *} \operatorname{Tor}^p_{\mathcal{O}_{\widetilde{D}_i}} \Big(\operatorname{Tor}^q_{\overline{f}_i^{\,-1}\mathcal{O}_D} \big(\overline{f}_i^{\,-1}\mathcal{F}, \mathcal{O}_{\widetilde{D}_i}\big), \mathcal{L} \Big)$$
$$E_\infty^{p,q} = \operatorname{Gr}^p \operatorname{Tor}^{p+q}_{i_{\widetilde{D}*}\overline{f}^{\,-1}\mathcal{O}_D} \big(i_{\widetilde{D}*}\overline{f}^{\,-1}\mathcal{F}, i_{\widetilde{D}_i *}\mathcal{L} \big)$$

This gives in $K_{\widetilde{D}_i}(X)$ the equality

$$\sum_{i\geq 0} (-1)^i \big[\operatorname{Tor}^i_{i_{\widetilde{D}*}\overline{f}^{\,-1}\mathcal{O}_D} \big(i_{\widetilde{D}*}\overline{f}^{\,-1}\mathcal{F}, i_{\widetilde{D}_i *}\mathcal{L} \big) \big] \simeq i_{\widetilde{D}_i *}\big(\overline{f}_i^{\,\dagger}[\mathcal{F}] \,.\, [\mathcal{L}] \big).$$

Therefore, $\Gamma([i_{\widetilde{D}_i *}\mathcal{L}]) = i_{\widetilde{D}_i *}\big(\overline{f}_i^{\,\dagger}[\mathcal{F}]\,.\,[\mathcal{L}]\big)$. This formula is valid if we replace $\mathcal{L}$ by any element of $K(\widetilde{D}_i)$. This yields

$$f^{\,\dagger}[i_{\widetilde{D}*}\mathcal{F}] = \sum_{i=1}^N i_{\widetilde{D}_i *}\big(\overline{f}_i^{\,\dagger}[\mathcal{F}]\,.\,\overline{u}_i\big) = \sum_{i=1}^N \overline{f}_i^{\,\dagger}[\mathcal{F}]\centerdot_{\widetilde{D}_i} u_i.$$

$\square$

7.4. Analytic K-theory with support in a divisor with simple normal crossing. Let X be a smooth complex compact manifold and D be a reduced divisor with simple normal crossing. The branches of D will be denoted by $D_1, \dots, D_N$ and $D_{ij} = D_i \cap D_j$. By a dévissage argument, the canonical map from $\bigoplus_{i=1}^n K_{D_i}(X)$ to $K_D(X)$ is surjective. If x belongs to $K_{D_{ij}}(X)$, then the element $(x, -x)$ of $K_{D_i}(X) \oplus K_{D_j}(X)$ is in the kernel of this map. We will show that this kernel is generated by these elements:

Proposition 7.8. *There is an exact sequence:*

$$\textstyle\bigoplus_{i<j} K_{D_{ij}}(X) \longrightarrow \bigoplus_i K_{D_i}(X) \longrightarrow K_D(X) \longrightarrow 0.$$

Proof. We will deal with the exact sequence

$$\textstyle\bigoplus_{i<j} K(D_{ij}) \longrightarrow \bigoplus_i K(D_i) \longrightarrow K(D) \longrightarrow 0.$$

By the dévissage theorem 7.2, this sequence is isomorphic to the initial one. We proceed by induction on the number N of the branches of D. Let D' be the divisor whose branches are $D_1, \dots, D_{N-1}$. We have a complex

$$(*) \qquad K(D' \cap D_N) \longrightarrow K(D') \oplus K(D_N) \xrightarrow{\ \pi\ } K(D) \longrightarrow 0$$

where the first map is given by $\alpha \longmapsto (\alpha, -\alpha)$. Let us verify that this complex is exact. Consider the map $\psi : \mathbb{Z}[\operatorname{coh}(D)] \longrightarrow K(D') \oplus K(D_N) \,/\, K(D' \cap D_N)$ defined by $\psi(\mathcal{F}) = [i_{D'}^*\mathcal{F}] + [\mathcal{I}_{D'}\mathcal{F}]$. Remark that $\mathcal{I}_{D'}$ is a sheaf of $\mathcal{O}_{D_N}$-modules, namely the sheaf of ideals of $D' \cap D_N$ in D_N extended to D by zero. Let us show that ψ can be defined in K-theory.
We consider an exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ of coherent sheaves on D. Let us define the sheaf $\mathcal{N}$ by the exact sequence

$$0 \longrightarrow \mathcal{N} \longrightarrow i_{D'}^*\mathcal{F} \longrightarrow i_{D'}^*\mathcal{G} \longrightarrow i_{D'}^*\mathcal{H} \longrightarrow 0.$$

It is clear that $\mathcal{N}$ is a sheaf of $\mathcal{O}_{D'}$-modules with support in $D' \cap D_N$. Furthermore, $[i^*_{D'}\mathcal{G}] - [i^*_{D'}\mathcal{F}] - [i^*_{D'}\mathcal{H}] = -[\mathcal{N}]$ in $K(D')$. Let us consider the following exact sequence of complexes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{I}_{D'}\mathcal{F} \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & i^*_{D'}\mathcal{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{D'}\mathcal{G} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & i^*_{D'}\mathcal{G} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_{D'}\mathcal{H} \arrow[r] & \mathcal{H} \arrow[r] & i^*_{D'}\mathcal{H} \arrow[r] & 0
\end{tikzcd}$$

Let $\mathcal{C}$ be the first complex, that is the first column of the diagram above. It is a complex of $\mathcal{O}_{D_N}$-modules. If we denote by $\mathcal{H}^k(\mathcal{C})$, $0 \leq k \leq 2$, the cohomology sheaves of $\mathcal{C}$, we have the long exact sequence

$$0 \longrightarrow \mathcal{H}^0(\mathcal{C}) \longrightarrow 0 \longrightarrow \mathcal{N} \longrightarrow \mathcal{H}^1(\mathcal{C}) \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{H}^2(\mathcal{C}) \longrightarrow 0.$$

Since $\mathcal{H}^1(\mathcal{C})$ is a sheaf of $\mathcal{O}_{D_N}$-modules, $\mathcal{N}$ is also a sheaf of $\mathcal{O}_{D_N}$-modules. Therefore, $\mathcal{N}$ is a sheaf of $\mathcal{O}_{D'\cap D_N}$-modules.

In $K(D_N)$ we have $[\mathcal{I}_{D'}\mathcal{F}] - [\mathcal{I}_{D'}\mathcal{G}] + [\mathcal{I}_{D'}\mathcal{H}] = [\mathcal{H}^0(\mathcal{C})] - [\mathcal{H}^1(\mathcal{C})] + [\mathcal{H}^2(\mathcal{C})] = -[\mathcal{N}]$. Thus $\psi(\mathcal{F}) - \psi(\mathcal{G}) + \psi(\mathcal{H}) = ([\mathcal{N}], -[\mathcal{N}]) = 0$ in the quotient.

If $\mathcal{F}$ belongs to $K(D)$, then $[\mathcal{F}] = [i^*_{D'}\mathcal{F}] + [\mathcal{I}_{D'}\mathcal{F}]$ in $K(D)$. This means that $\pi \circ \psi = \mathrm{id}$. We consider now $\mathcal{H}$ in $K(D')$ and $\mathcal{K}$ in $K(D_N)$. Then

$$\psi(\pi(\mathcal{H},\mathcal{K})) = ([i^*_{D'}\mathcal{H}] + [i^*_{D'}\mathcal{K}]) \oplus ([\mathcal{I}_{D'}\mathcal{H}] + [\mathcal{I}_{D'}\mathcal{K}]) = ([\mathcal{H}] + [\mathcal{K}_{|D'\cap D_N}]) \oplus [\mathcal{I}_{D'\cap D_N}\mathcal{K}].$$

Remark that $[\mathcal{I}_{D'\cap D_N}\mathcal{K}] = [\mathcal{K}] - [\mathcal{K}_{|D'\cap D_N}]$ in $K\big(D_N\big)$. Thus

$$([\mathcal{H}] + [\mathcal{K}_{|D'\cap D_N}]) \oplus ([\mathcal{K}] - [\mathcal{K}_{|D'\cap D_N}]) = [\mathcal{H}] \oplus [\mathcal{K}] \quad \text{modulo } K(D' \cap D_N),$$

so that $\psi \circ \pi = \mathrm{id}$. This proves that $(*)$ is exact.

We can now use the induction hypothesis with D'. We obtain the following diagram, where the columns as well as the first line are exact:

$$\begin{tikzcd}
0 & 0 & & \\
K(D' \cap D_N) \arrow[u] \arrow[r, "r"] & K(D') \oplus K(D_N) \arrow[u] \arrow[r, "u"] & K(D) \arrow[r] & 0 \\
\bigoplus_{i<N} K(D_{iN}) \arrow[u, "q"] \arrow[r, "s"] & \bigoplus_{i<N} K(D_i) \oplus K(D_N) \arrow[u, "p"] & & \\
& \bigoplus_{i<j<N} K(D_{ij}) \arrow[u, "t"] & &
\end{tikzcd}$$

The map $\bigoplus_i K(D_i) \xrightarrow{\pi} K(D)$ is clearly onto. Let α be an element of $\bigoplus_i K(D_i)$ such that $\pi(\alpha) = 0$. Then $u(p(\alpha)) = 0$, so that there exists β such that $r(\beta) = p(\alpha)$. There exists γ such that $q(\gamma) = \beta$. Then $p(\alpha - s(\gamma)) = p(\alpha) - r(q(\gamma)) = 0$. So there exists δ such that $\alpha = s(\gamma) + t(\delta)$. It follows that α is in the image of $\bigoplus_{i<j} K(D_{ij})$. Hence we have the exact sequence

$$\bigoplus_{i<j} K(D_{ij}) \xrightarrow{s+t} \bigoplus_i K(D_i) \xrightarrow{\pi} K(D) \longrightarrow 0,$$

which finishes the proof. ∎

Bibliographie

[An-Ga] V. Ancona, B. Gaveau, Differential Forms on Singular Varieties, Chapman and Hall/CRC, (2006).

[At] M. F. Atiyah, Complex analytic connections in fiber bundles, Trans. Amer. Math. Soc., *85*, (1957), 181–207.

[At-Hi] M. F. Atiyah, F. Hirzebruch, The Riemann-Roch theorem for Analytic embeddings, Topology, *1*, (1962), 151–166.

[Bei] A. A. Beilinson, Coherent sheaves on $\mathbb{P}^n$ and problems in linear algebra, Funkt. Analiz. Appl., *12(3)*, (1978), 68–69.

[Ber] A. Berthomieu, Proof of Nadel's conjecture and direct image for relative K-theory, Bull. Soc. math. France, *130-2*, (2002), 253–307.

[Bl] S. Bloch, Semi-regularity and de Rham cohomology, Invent. Math., *17*, (1972), 51–66.

[Bo-Se] A. Borel, J.-P. Serre, Le théorème de Riemann-Roch (d'après Grothendieck), Bull. Soc. Math. France, *86*, (1958), 97–136.

[Bott] R. Bott, Lectures on characteristic classes and foliations, in *Lectures on Algebraic and Differential Topology*, Lectures Notes in Math, *279*, Springer, (1972).

[Br] J.-L. Brylinski, Comparison of the Beilinson-Chern classes with the Chern-Cheeger-Simons classes, in *Advances in Geometry*, Prog. Nat., *172*, Birkhäuser, Boston, (1999), 95–105.

[Ch-Si] J. Cheeger, J. Simons, Differential characters and geometric invariants, in *Geometry and Topology* (College Park 83-84), Lecture Notes in Math., *1167*, Springer, (1985), 50–80.

[Ei] D. Eisenbud, Commutative Algebra, Springer, (1995).

[Es] H. Esnault, Characteristic classes of flat bundles, Topology, *27-3*, (1988), 323–352.

[Es-Vi] H. Esnault, E. Viehweg, Deligne-Beilinson Cohomology, in *Beilinson's conjecture on special values of L-functions* (Rapoport, Schappacher & Schneider, eds.) Perspec. Math., Vol. 4, Academic Press, (1988), 43–91.

[EZZ] F. El Zein, S. Zucker, Extendability of normal functions associated to algebraic cycles, in *Topics in Transcendental Algebraic Geometry*, Annals of Math. Studies, *106*, Princeton University Press, (1984), 269–288.

[Fe] H. Federer, Geometric Measure Theory, Springer, (1969).

[Fu] W. Fulton, Intersection Theory, Springer, (1984).

[Ga] P. Gajer, Geometry of Deligne cohomology, Inv. Math., Springer *127*, (1997), 155–207.

[Gi-So] H. Gillet, C. Soulé, Arithmetic Chow groups and differential characters, in *Algebraic K-theory: Connections with Geometry and Topology,* Ed. J. F. Jardine, V. P. Snaith, Kluwer, (1989), 29–68.

[Gra] H. Grauert, On Levi's problem and the imbedding of real-analytic manifolds, Ann. Math., Princeton, *68*, (1958), 460–472.

[Gra-Re] H. Grauert, R. Remmert, Coherent Analytic Sheaves, Springer, (1984).

[Gre] H. I. Green, Chern classes for coherent sheaves, Ph.D. Thesis, University of Warwick, January 1980.

[Gri-Ha] P. Griffiths, J. Harris Principles of Algebraic Geometry, John Wiley and Sons, (1978).

[Gro 1] A. Grothendieck, La théorie des classes de Chern, Bull. Soc. Math. France, *86*, (1958), 137–154.

[Gro 2] A. Grothendieck, Éléments de géométrie algébrique III, étude cohomologique des faisceaux cohérents, *Publications Mathématiques de l'I.H.É.S.*, *11*, (1961).

[Gro 3] A. Grothendieck, On the de Rham cohomology of algebraic varieties, *Publications Mathématiques de l'I.H.É.S.*, *29*, (1966) 95–103.

[EGA IV] A. Grothendieck, J. Dieudonné, Étude locale des schémas et des morphismes de schémas, *Publications Mathématiques de l'I.H.É.S.,* (1967).

[Gr-Ra] L. Gruson, M. Raynaud, Critères de platitude et de projectivité, Inv. Math., *13*, (1971), 1–89.

[Hiro 1] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math., *79*, (1964), 109–326.

[Hiro 2] H. Hironaka, Flattening Theorem in Complex-Analytic Geometry, Amer. J. of Math., *97-2*, (1975), 503–547.

[Hirz] F. Hirzebruch, Topological methods in algebraic geometry, Springer, (1966).

[Ka-Sc] M. Kashiwara, P. Schapira, Sheaves on manifolds, Springer, (2002).

[Ill] L. Illusie, Complexe cotangent et déformations I, Lecture Notes in Math., Springer, *239*, (1971).

[Ki] J. King, The currents defined by analytic varieties, Acta Math., *127*, (1971), 185–220.

[Kn-Mu] F. F. Knudsen, D. Mumford, The projectivity of moduli spaces of stable curves I: Preliminaries on 'det' and 'Div', Math. Scand., *39*, (1976), 19–55.

[LMS] A. T. Lascu, D. Mumford, D. B. Scott, The self-intersection formula and the "formule-clef", Proc. Cambridge Phil. Soc., *78*, (1975), 117–123.

[Na] A. Nadel, Invariants for holomorphic vector bundles, Math. Ann., *309*, (1997), 37–52.

[OB-To-To 1] N. R. O'Brian, D. Toledo, Y. L. L. Tong, A trace map and characteristic classes for coherent sheaves, Amer. J. Math., *103*, (1981), 253–271.

[OB-To-To 2] N. R. O'Brian, D. Toledo, Y. L. L. Tong, Grothendieck-Riemann-Roch for complex manifolds, Bull. Amer. Math. Soc., *5-2*, (1981), 182–184.

[OB-To-To 3] N. R. O'Brian, D. Toledo, Y. L. L. Tong, A Grothendieck-Riemann-Roch formula for maps of complex manifolds, Math. Ann., *271*, (1985), 493–526.

[Qui] D. Quillen, Higher algebraic K-theory I, in *Higher algebraic K-theory I,* Lecture Notes in Math., 431, Springer, (1972).

[Ro] H. Rossi, Picard variety of an isolated singular point, Rice Univ. Studies, , *54-4*, (1968), 63–73.

[ScVe] S. Schröer, G. Vezzosi, Existence of vector bundles and global resolutions for singular surfaces, Compos. Math., *140-3*, (2004), 717–728.

[Sch] H. W. Schuster, Locally free resolutions of coherent sheaves on surfaces, J. Reine Angew. Math., *337*, (1982), 159–165.

[SGA $4\frac{1}{2}$] P. Deligne, Séminaire de Géométrie Algébrique du Bois Marie 1964/65 SGA $4\frac{1}{2}$, *Cohomologie étale,* Lecture Notes in Math., *569*, Springer, (1976).

[SGA 5] A. Grothendieck, Séminaire de Géométrie Algébrique du Bois Marie 1965/66, SGA 5, *Cohomologie l-adique et fonctions L,* Lecture Notes in Math., *589*, Springer, (1977).

[SGA 6] P. Berthelot, A Grothendieck, L. Illusie, Séminaire de Géométrie Algébrique du Bois Marie 1966/67, SGA 6, *Théorie des Intersections et Théorème de Riemann-Roch,* Lecture Notes in Math., *225*, Springer, (1971).

[To-To] D. Toledo, Y. L. L. Tong, Green's theory of Chern classes and the Riemann-Roch formula, Contemp. Math., *58-I*, (1986), 261–275.

[Vo 1] C. Voisin, Théorie de Hodge et géométrie algébrique complexe, Société Mathématique de France, (2002).

[Vo 2] C. Voisin, A counterexample to the Hodge conjecture extended to Kähler varieties, Int. Math. Res. Notices, *20*, (2002), 1057–1075.

[Zu] S. Zucker, The Cheeger-Simons invariant as a Chern class, in *Algebraic Analysis, Geometry and Number Theory (Baltimore 1988),* Johns Hopkins Univ. Press, Baltimore, (1989), 397–417.

CHAPITRE 3

Compléments sur la cohomologie de Deligne

1. Introduction

Le but de cette section est d'expliquer la construction du morphisme de Gysin en cohomologie de Deligne. Si X est une varieté complexe et $Z \subseteq X$ un cycle localement intersection complète de codimension d, on dispose d'une classe de cycle canonique $\{Z\}$ dans $H^d_Z\big(X, \Omega^d_X\big)$ [**Gr**], [**Ka-Sc**]. Il est montré dans [**Bl**] que le morphisme naturel $\mathbb{H}^{2d}_Z\big(X, \Omega^{\bullet \geqslant d}_X\big) \longrightarrow H^d_Z\big(X, \Omega^d_X\big)$ est injectif et que $\{Z\}$ est élément de $\mathbb{H}^{2d}_Z\big(X, \Omega^{\bullet \geqslant d}_X\big)$. Cette classe, notée $\big\{Z\big\}_{\mathrm{Bl}}$, est la classe de Bloch de Z. Le morphisme naturel

$$\mathbb{H}^{2d}_Z\big(X, \Omega^{\bullet \geqslant d}_X\big) \longrightarrow \mathbb{H}^{2d}_Z\big(X, \Omega^{\bullet}_X\big) \simeq H^{2d}_Z(X, \mathbb{C})$$

envoie $\big\{Z\big\}_{\mathrm{Bl}}$ sur $(2i\pi)^d\big\{Z\big\}_{\mathrm{top}}$, où $\big\{Z\big\}_{\mathrm{top}}$ est la classe de cycle topologique de Z. La suite exacte

$$0 \longrightarrow \mathbb{H}^{2d}_Z\big(X, \mathbb{Z}_{D,X}(d)\big) \longrightarrow H^{2d}_Z(X, \mathbb{Z}) \oplus \mathbb{H}^{2d}_Z\big(X, \Omega^{\bullet \geqslant d}_X\big) \xrightarrow{(2i\pi)^d, -1} H^{2d}_Z(X, \mathbb{C}) \longrightarrow 0$$

montre que $\big(\big\{Z\big\}_{\mathrm{top}}, \big\{Z\big\}_{\mathrm{Bl}}\big)$ provient d'un élément canonique dans $\mathbb{H}^{2d}_Z\big(X, \mathbb{Z}_{D,X}(d)\big)$. L'image de cet élément dans $H^{2d}_{\mathrm{Del}}\big(X, \mathbb{Z}(d)\big)$, notée $\big\{Z\big\}_D$, est la classe de Deligne de Z.

Plus généralement, si $f : X \longrightarrow Y$ est un morphisme holomorphe propre entre variétés complexes et si $d = \dim Y - \dim X$, il existe un morphisme de Gysin canonique

$$f_* \colon H^i_{\mathrm{Del}}\big(X, \mathbb{Z}(p)\big) \longrightarrow H^{i+2d}_{\mathrm{Del}}\big(Y, \mathbb{Z}(p+d)\big)$$

compatible avec les morphismes de Gysin usuels

$$f_* \colon H^i(X, \mathbb{Z}) \longrightarrow H^{i+2d}(Y, \mathbb{Z}) \quad \text{et} \quad f_* \colon \mathbb{H}^i\big(X, \Omega^{\bullet \geqslant p}_X\big) \longrightarrow \mathbb{H}^{i+2d}\big(Y, \Omega^{\bullet \geqslant p+d}_Y\big).$$

Cette construction est esquissée dans [**EZZ**], l'idée étant de réaliser le morphisme f_* au niveau des complexes. Pour ceci, on résout le faisceau constant $\mathbb{Z}_X$ par un sous-complexe de de Rham des courants sur X qui est le complexe des courants entiers [**Ki**], [**Fe**]. Cette résolution est également utilisée dans [**Gi-So**] pour construire un morphisme d'image directe par une submersion dans le cadre des caractères différentiels de Cheeger-Simons.

L'organisation et les références utilisées pour ce chapitre sont les suivantes :

Définition de la cohomologie de Deligne.
Références [**Es-Vi**], [**Vo**].

La structure d'anneau. On décrit le cup-produit pour les complexes de Deligne en suivant [**Es-Vi**] et on étend ces formules à d'autres complexes quasi-isomorphes au complexe de Deligne.

Classe de Bloch
Références [**Bl**], [**Ka-Sc**].

La classe de cycle en cohomologie de Deligne Dans cette section, si Z est une sous-variété lisse (ou localement intersection complète) de codimension d d'une variété complexe compacte X, on calcule le complexe $\mathcal{R}\Gamma_Z(\mathbb{Z}_{D,X}(p))$. Ceci permet de retrouver la classe de cycle de Z en cohomologie de Deligne dans $H^{2d}_{\mathrm{Del}}(X,\mathbb{Z}(d))$ [**EZZ**], [**Es-Vi**], mais également de construire un morphisme de Gysin $i_{Z*}\colon H^i_{\mathrm{Del}}(Z,\mathbb{Z}(p)) \longrightarrow H^{i+2d}_{\mathrm{Del}}(X,\mathbb{Z}(p+d))$ sans recourir aux courants entiers.

Courants normaux, courants rectifiables, courants entiers. On rappelle des notions de base de la théorie géométrique de la mesure, la référence étant le livre de Federer [**Fe**]. Le but est d'expliquer la manière dont est construite la résolution du faisceau constant $\mathbb{Z}_X$ par les faisceaux de courants entiers.

Morphisme de Gysin en cohomologie de Deligne. En utilisant les résultats de la section 6, on construit le morphisme de Gysin f_* comme indiqué dans [**EZZ**]. On montre la compatibilité entre cette construction et la construction de la section 5 dans le cas où f est l'injection i_Z d'un cycle lisse Z de X. Enfin, en utilisant les formules de cup-produit établies dans la section 3, on montre la formule de projection ainsi qu'une formule de commutation entre image directe et image inverse dans un cas sans excès.

Remerciements. Je tiens à remercier Thierry de Pauw et Pierre Schapira pour m'avoir patiemment expliqué les courants rectifiables pour le premier et les faisceaux $B_{Z|X}$ ainsi que la théorie des hyperfonctions pour le second.

2. Définition de la cohomologie de Deligne

Soit X une variété complexe compacte et lisse, et soit p un entier.

DÉFINITION 2.1. *Le* complexe de Deligne entier de X de niveau p *est le complexe de faisceaux* $\mathbb{Z}_{D,X}(p)$ *suivant*

$$0 \longrightarrow \mathbb{Z}_X \xrightarrow{(2i\pi)^p} \mathcal{O}_X \xrightarrow{d} \Omega^1_X \longrightarrow \cdots \longrightarrow \Omega^{p-1}_X$$

où $\mathbb{Z}_X$ est en degré zéro. On définit de même le complexe de Deligne rationnel de X de niveau p en remplaçant $\mathbb{Z}_X$ par $\mathbb{Q}_X$.

DÉFINITION 2.2. *Les groupes de cohomologie de Deligne entière et rationnelle de X sont définis par les formules*

$$\begin{cases} H^k_{Del}(X,\mathbb{Z}(p)) = \mathbb{H}^k(X,\mathbb{Z}_{D,X}(p)) \\ H^k_{Del}(X,\mathbb{Q}(p)) = \mathbb{H}^k(X,\mathbb{Q}_{D,X}(p)). \end{cases}$$

On pose alors

$$\begin{cases} H^*_{Del}(X,\mathbb{Z}) = \bigoplus_{k,p} H^k_{Del}(X,\mathbb{Z}(p)) \\ H^*_{Del}(X,\mathbb{Q}) = \bigoplus_{k,p} H^k_{Del}(X,\mathbb{Q}(p)). \end{cases}$$

On a $\mathbb{Z}_{D,X}(0) = \mathbb{Z}_X$, et $\mathbb{Z}_{D,X}(1) = \ \mathbb{Z}_X \xrightarrow{(2i\pi)} \mathcal{O}_X$ est quasi-isomorphe à $\mathcal{O}_X^*[-1]$ par la suite exacte exponentielle. On en déduit le fait fondamental suivant :

$$H^2\big(X, \mathbb{Z}(1)\big) \simeq H^1\big(X, \mathcal{O}_X^*\big) \simeq \mathrm{Pic}(X).$$

On ne considère pour les classes de Chern que les groupes $H^{2p}_{\mathrm{Del}}\big(X, \mathbb{Q}(p)\big)$. Cependant, d'autres groupes de cohomologie de Deligne ont un intérêt géométrique. Par exemple, Deligne établit un isomorphisme entre $H^2_{\mathrm{Del}}\big(X, \mathbb{Z}(2)\big)$ et les fibrés holomorphes en droites munis d'une connexion holomorphe. L'isomorphisme est construit de la manière suivante : $\mathbb{Z}_{D,X}(2)$ est quasi-isomorphe à $\big(\ \mathcal{O}_X^\times \xrightarrow[d\log]{} \Omega^1_X\ \big)[-1]$. Un 1-cocycle de Čech est alors donné par des fonctions holomorphes g_{ij} inversibles sur U_{ij} et des 1-formes holomorphes ω_i sur U_i telles que $\frac{dg_{ij}}{g_{ij}} = \omega_i - \omega_j$. Les g_{ij} définissent un fibré en droites holomorphe avec une trivialisation sur chaque U_i, les ω_i sont alors les 1-formes de connexion associées à la trivialisation. La suite exacte

$$0 \longrightarrow \Omega^1_X[-2] \longrightarrow \mathbb{Z}_{D,X}(2) \longrightarrow \mathbb{Z}_{D,X}(1) \longrightarrow 0$$

fournit la suite longue :

$$H^0\big(X, \Omega^1_X\big) \longrightarrow H^2\big(X, \mathbb{Z}_{D,X}(2)\big) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\mathrm{At}} H^1\big(X, \Omega^1_X\big),$$

où la dernière flèche est la classe d'Atiyah. On retrouve ainsi le fait qu'un fibré en droites holomorphe peut être muni d'une connexion holomorphe si et seulement si sa classe d'Atiyah est nulle. La première flèche associe à une 1-forme holomorphe globale ω le fibré trivial avec la connexion $d + \omega\wedge$.

La cohomologie de Deligne permet de concilier des données cohomologiques holomorphes et des données topologiques. Le prototype de ce phénomène est la notion de classe de Hodge : $\mathrm{Hdg}^{2p}(X, \mathbb{Z})$ est l'ensemble des éléments de $H^{2p}(X, \mathbb{Z})$ qui peuvent être représentés comme classe de $H^{2p}(X, \mathbb{C})$ par une forme fermée de type (p, p).

LEMME 2.3. *Si X est une variété Kählérienne, on a une suite exacte*

$$0 \longrightarrow J^{2p-1}(X) \longrightarrow H^{2p}_{\mathrm{Del}}\big(X, \mathbb{Z}(p)\big) \longrightarrow \mathrm{Hdg}^{2p}(X, \mathbb{Z}) \longrightarrow 0$$

où

$$J^{2p-1}(X) = H^{2p-1}(X, \mathbb{C}) \Big/ \big(H^{2p-1}(X, \mathbb{Z}) + F^p H^{2p-1}(X, \mathbb{C})\big)$$

est la $p^{\text{ième}}$ jacobienne intermédiaire.

DÉMONSTRATION. La suite exacte

$$0 \longrightarrow \Omega_X^{\bullet \leqslant p-1}[-1] \longrightarrow \mathbb{Z}_{D,X}(p) \longrightarrow \mathbb{Z}_X \longrightarrow 0$$

fournit la suite longue

$$H^{2p-1}(X, \mathbb{Z}) \longrightarrow \mathbb{H}^{2p-1}\big(X, \Omega_X^{\bullet \leqslant p-1}\big) \longrightarrow H^{2p}_{\mathrm{Del}}\big(X, \mathbb{Z}(p)\big) \longrightarrow H^{2p}(X, \mathbb{Z}) \longrightarrow \mathbb{H}^{2p-1}\big(X, \Omega_X^{\bullet \leqslant p-1}\big).$$

Comme X est une variété Kählérienne, $\mathbb{H}^i\big(X, \Omega_X^{\bullet \geqslant p}\big) = F^p H^i(X, \mathbb{C}) \hookrightarrow H^i(X, \mathbb{C})$ et ainsi

$$\mathbb{H}^i\big(X, \Omega_X^{\bullet \leqslant p-1}\big) = H^i(X, \mathbb{C}) \Big/ F^p H^i(X, \mathbb{C}),$$

d'où le résultat. □

Les groupes $H^{2p}_{\mathrm{Del}}\big(X,\mathbb{Z}(p)\big)$ sont donc plus riches que les groupes $\mathrm{Hdg}^{2p}(X,\mathbb{Z})$ car ils contiennent des informations secondaires du type Abel-Jacobi.

La définition initiale du complexe de Deligne est la plus simple que l'on puisse donner, mais il est souvent plus agréable sur le plan théorique de le voir comme un cône :

LEMME 2.4. *$\mathbb{Z}_{D,X}(p)$ est quasi-isomorphe au cône du morphisme $\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant p} \xrightarrow[-(2i\pi)^p,\, i]{} \Omega_X^{\bullet}$ décalé de -1.*

Rappelons pour fixer les conventions de signe que si $f\colon A \longrightarrow B$ est un morphisme de complexes, le mapping cone de f est le complexe $\mathrm{Mc}(f) = B \oplus A[1]$, muni de la différentielle

$$\delta(b_k, a_{k+1}) = \big(\delta b_k + (-1)^k f(a_{k+1}), \delta a_{k+1}\big).$$

On a une suite exacte $0 \longrightarrow B \longrightarrow \mathrm{Mc}(f) \longrightarrow A[1] \longrightarrow 0$ donnée par les deux applications $b \longmapsto (b,0)$ et $(b,a) \longmapsto a$.

DÉMONSTRATION. Le complexe $\mathrm{Mc}\Big(\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant p} \xrightarrow[-(2i\pi)^p,\, i]{} \Omega_X^{\bullet}\Big)[-1]$ s'écrit

$$\{0\} \oplus \mathbb{Z}_X \longrightarrow \mathcal{O}_X \oplus \{0\} \longrightarrow \Omega_X^1 \oplus \{0\} \longrightarrow \cdots \longrightarrow \Omega_X^{p-2} \oplus \{0\} \longrightarrow \Omega_X^{p-1} \oplus \Omega_X^p \longrightarrow \cdots$$

où $\mathbb{Z}_X$ est en degré 0 et les différentielles sont données par

$$\begin{cases} \delta(0,m) = \big((2i\pi)^p m, 0\big) & \\ \delta(\beta, 0\,) = (d\beta, 0) & \text{si } \mathrm{d}^\circ(\beta) \leq p-2 \\ \delta(\beta,\alpha) = \big(d\beta + (-1)^{\mathrm{d}^\circ(\beta)}\alpha, d\alpha\big) & \text{si } \mathrm{d}^\circ(\beta) \geq p-1. \end{cases}$$

On définit alors deux morphismes de complexes u_p et v_p

$$\mathbb{Z}_{D,X} \underset{v_p}{\overset{u_p}{\rightleftarrows}} \mathrm{Mc}\Big(\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant p} \longrightarrow \Omega_X^{\bullet}\Big)[-1]$$

par

$$u_p(x) = \begin{cases} (0,x) & \text{si } \mathrm{d}^\circ(x) = 0 \\ (x,0) & \text{si } 0 < \mathrm{d}^\circ(x) < p \\ (x, (-1)^p dx) & \text{si } \mathrm{d}^\circ(x) = p \end{cases} \qquad \text{et} \qquad v_p(\beta,\alpha) = \begin{cases} \alpha & \text{si } \mathrm{d}^\circ(\alpha) = 0 \\ \beta & \text{si } \mathrm{d}^\circ(\beta) \leq p-1 \\ 0 & \text{sinon.} \end{cases}$$

On a évidemment $v_p \circ u_p = \mathrm{id}$. Montrons que u_p et v_p sont des quasi inverses. Il suffit de montrer que u_p induit un isomorphisme en cohomologie, le seul degré à vérifier est le degré p. Soit (β,α) un élément de $\Omega_X^{p-1} \oplus \Omega_X^p$ tel que $d\alpha = 0$ et $d\beta + (-1)^{p-1}\alpha = 0$. Alors $(\beta,\alpha) = \big(\beta, (-1)^p d\beta\big) = u_p(\beta)$, donc $[u_p]$ est surjectif. Si β appartient à Ω_X^{p-1} et si $[u_p](\beta) = 0$, alors $\beta = d\gamma$, donc $[u_p]$ est injectif. $\square$

COROLLAIRE 2.5. *On a une suite longue*

$$\cdots \longrightarrow H^{k-1}(X,\mathbb{C}) \longrightarrow H^k_{\mathrm{Del}}\big(X,\mathbb{Z}(p)\big) \longrightarrow \mathbb{H}^k\big(X, \Omega_X^{\bullet \geqslant p}\big) \oplus H^k(X,\mathbb{Z}) \longrightarrow H^k(X,\mathbb{C}) \longrightarrow \cdots$$

3. La structure d'anneau

Dans cette section, on construit en détail le cup-produit sur les groupes de cohomologie de Deligne, en donnant toutes les formules explicites selon les représentants des complexes utilisés.

Rappelons pour fixer les conventions de signe que si R est un anneau commutatif et A et B sont deux complexes bornés à gauche de R-modules, le complexe $A \otimes_R B$ est défini par

$$\big(A \otimes_R B\big)_k = \bigoplus_{i+j=k} A_i \otimes_R B_j$$

et la différentielle est définie par

$$\delta(a \otimes b) = \delta a \otimes b + (-1)^{\mathrm{d}^\circ(a)} a \otimes \delta b.$$

On a alors un isomorphisme de complexes $A \otimes_R B \simeq B \otimes_R A$ donné par

$$a \otimes b \longmapsto (-1)^{\mathrm{d}^\circ(a)\mathrm{d}^\circ(b)} b \otimes a.$$

Pour éviter d'écrire de nombreux facteurs $(2i\pi)^k$, on raisonnera sur le complexe $\mathbb{Z}'_{D,X}(p)$ suivant :

$$\mathbb{Z}_X \xrightarrow{\ i\ } \mathcal{O}_X \xrightarrow{\ d\ } \Omega^1_X \longrightarrow \cdots \longrightarrow \Omega^{p-1}_X$$

qui est isomorphe à $\mathbb{Z}_{D,X}(p)$.

Le cup-produit en cohomologie de Deligne est défini de la manière suivante : soit

$$\psi = \psi_{pq} \colon \mathbb{Z}'_{D,X}(p) \otimes_{\mathbb{Z}} \mathbb{Z}'_{D,X}(q) \longrightarrow \mathbb{Z}'_{D,X}(p+q)$$

défini par

$$\psi(x \otimes y) = \begin{cases} xy & \text{si } \mathrm{d}^\circ(x) = 0 \\ x \wedge dy & \text{si } \mathrm{d}^\circ(x) > 0 \quad \text{et } \mathrm{d}^\circ(y) = q \\ 0 & \text{sinon} \end{cases}$$

Vérifions que ψ est bien un morphisme de complexes. On étudie les différents cas

1. Supposons que $\mathrm{d}^\circ(x) = 0$. Alors $\psi(x \otimes y) = xy$.

(i) Si $\mathrm{d}^\circ(y) = q$,

$$\delta\big(\psi(x \otimes y)\big) = x\,dy \quad \text{et} \quad \psi(\delta x \otimes y) + \psi(x \otimes \delta y) = x\,dy \text{ car } \delta y = 0.$$

(ii) Si $\mathrm{d}^\circ(y) < q$,

$$\delta\big(\psi(x \otimes y)\big) = x\,\delta y \quad \text{et} \quad \psi(\delta x \otimes y) + \psi(x \otimes \delta y) = \psi(x \otimes \delta y) = x\,\delta y.$$

2. Supposons que $\mathrm{d}^\circ(x) > 0$.

(i) Si $\mathrm{d}^\circ(y) = q$ et $\mathrm{d}^\circ(x) < p$,

$$\psi(x \otimes y) = x \wedge dy \quad \text{et} \quad \delta\big(\psi(x \otimes y)\big) = dx \wedge dy$$

$$\psi(\delta x \otimes y) + (-1)^{\mathrm{d}^\circ(x)}\psi(x \otimes \delta y) = \psi(\delta x \otimes y) = dx \wedge dy.$$

(ii) Si $\mathrm{d}^\circ(y) = q$ et $\mathrm{d}^\circ(x) = p$, $\delta\big(\psi(x \otimes y)\big) = 0$, $\delta x = 0$ et l'autre terme vaut aussi 0.

(iii) Si $\mathrm{d}^\circ(y) < q$, $\psi(x \otimes y) = 0$, $\psi(\delta x \otimes y) = 0$ et $\psi(x \otimes \delta y) = 0$ même si $\mathrm{d}^\circ(y) = q - 1$ car $x \wedge d(dy) = 0$.

LEMME 3.1. *Les deux morphismes*

$$\psi = \psi_{pq} : \mathbb{Z}'_{D,X}(p) \otimes_{\mathbb{Z}} \mathbb{Z}'_{D,X}(q) \longrightarrow \mathbb{Z}'_{D,X}(p+q) \qquad \textit{et}$$

$$\mathbb{Z}'_{D,X}(p) \otimes_{\mathbb{Z}} \mathbb{Z}'_{D,X}(q) \xrightarrow{\simeq} \mathbb{Z}'_{D,X}(q) \otimes_{\mathbb{Z}} \mathbb{Z}'_{D,X}(p) \xrightarrow{\psi_{qp}} \mathbb{Z}'_{D,X}(p+q)$$

sont homotopes.

DÉMONSTRATION. On définit un morphisme d'homotopie h par

$$h(x \otimes y) = \begin{cases} 0 & \text{si } \mathrm{d}^\circ(x) = 0 \text{ ou } \mathrm{d}^\circ(y) = 0 \\ (-1)^{\mathrm{d}^\circ(x)} x \wedge y & \text{sinon.} \end{cases}$$

Montrons que

$$-\psi_{pq}(x \otimes y) + (-1)^{\mathrm{d}^\circ(x)\mathrm{d}^\circ(y)} \psi_{qp}(y \otimes x) = \delta\big(h(x \otimes y)\big) + h\big(\delta(x \otimes y)\big).$$

On pose

$$A = -\psi_{pq}(x \otimes y) + (-1)^{\mathrm{d}^\circ(x)\mathrm{d}^\circ(y)}\, \psi_{qp}(y \otimes x) \qquad \text{et}$$
$$B = \delta\big(h(x \otimes y)\big) + h(\delta x \otimes y) + (-1)^{\mathrm{d}^\circ(x)} h(x \otimes \delta y).$$

On distingue les différents cas

1. $\mathrm{d}^\circ(x) > 0$ et $\mathrm{d}^\circ(y) > 0$. On a alors

$$B = (-1)^{\mathrm{d}^\circ(x)} \delta(x \wedge y) + (-1)^{\mathrm{d}^\circ(x)+1} \delta x \wedge y + x \wedge \delta y.$$

(i) Si $\mathrm{d}^\circ(x) < p$ et $\mathrm{d}^\circ(y) < q$, A est nul et

$$B = (-1)^{\mathrm{d}^\circ(x)} dx \wedge y - x \wedge dy - (-1)^{\mathrm{d}^\circ(x)} dx \wedge y + x \wedge dy = 0.$$

(ii) Si $\mathrm{d}^\circ(x) = p$ et $\mathrm{d}^\circ(y) < q$,

$$A = (-1)^{\mathrm{d}^\circ(x)\mathrm{d}^\circ(y)} y \wedge dx$$
$$B = (-1)^{\mathrm{d}^\circ(x)} d(x \wedge y) + x \wedge dy \qquad \text{car } \delta x = 0$$
$$= (-1)^{\mathrm{d}^\circ(x)} dx \wedge y = (-1)^{\mathrm{d}^\circ(x)} (-1)^{\mathrm{d}^\circ(x)(\mathrm{d}^\circ(y)-1)} y \wedge dx = (-1)^{\mathrm{d}^\circ(x)\mathrm{d}^\circ(y)} y \wedge dx$$

(iii) Si $\mathrm{d}^\circ(x) < p$ et $\mathrm{d}^\circ(y) = q$,

$$A = -x \wedge dy \quad \text{et} \quad B = (-1)^{\mathrm{d}^\circ(x)} d(x \wedge y) + (-1)^{\mathrm{d}^\circ(x)+1} dx \wedge y = -x \wedge dy.$$

2. $\mathrm{d}^\circ(x) = 0$ et $\mathrm{d}^\circ(y) > 0$. Alors $A = B = -xy$.

3. $\mathrm{d}^\circ(x) > 0$ et $\mathrm{d}^\circ(y) = 0$. Alors $A = B = yx$.

4. $\mathrm{d}^\circ(x) = 0$ et $\mathrm{d}^\circ(y) = 0$. Alors $A = -xy + yx = 0$ et $B = 0$.

□

Le morphisme ψ_{pq} induit un morphisme en cohomologie de Deligne

$$H^i_{\mathrm{Del}}(X, \mathbb{Z}(p)) \otimes_{\mathbb{Z}} H^j_{\mathrm{Del}}(X, \mathbb{Z}(p)) \longrightarrow H^{i+j}_{\mathrm{Del}}(X, \mathbb{Z}(p+q))$$

noté $\alpha \otimes_{\mathbb{Z}} \beta \longmapsto \alpha \cup \beta$. Le lemme 3.1 montre que $\alpha \cup \beta = (-1)^{ij} \beta \cup \alpha$.

Remarquons que les flèches $H^k_{\mathrm{Del}}\big(X,\mathbb{Z}(p)\big) \longrightarrow H^k_{\mathrm{Del}}(X,\mathbb{Z})$ et $H^k_{\mathrm{Del}}\big(X,\mathbb{Z}(p)\big) \xrightarrow{[\pi]} \mathbb{H}^k\big(\Omega_X^{\bullet\geqslant p}\big)$ sont des morphismes d'anneaux. Pour la deuxième flèche, il suffit d'écrire en se plaçant au niveau des complexes

$$\pi(x) = \begin{cases} (-1)^p dx_p & \text{en degré } p \\ 0 & \text{ailleurs,} \end{cases}$$

d'après la description explicite de u_p et v_p faite dans la preuve du lemme 2.4. Si x appartient à $\mathbb{Z}_{D,X}(p)$ et y appartient à $\mathbb{Z}_{D,X}(q)$,

$$\begin{aligned} \pi(x)\wedge\pi(y) &= (-1)^{p+q} dx_p \wedge dy_q = (-1)^{p+q} d(x_p \wedge dy_q) \\ &= (-1)^{p+q} d\big[\psi_{pq}(x\otimes y)\big]_{p+q} = \pi\big(\psi_{pq}(x\otimes y)\big). \end{aligned}$$

Vérifions enfin que le cup-produit est associatif.

Lemme 3.2. *On a*

$$\psi_{p+q,r}\big(\psi_{pq}\otimes \mathrm{id}_{\mathbb{Z}'_{D,X}(r)}\big) = \psi_{p,q+r}\big(\mathrm{id}_{\mathbb{Z}'_{D,X}(p)}\otimes\psi_{qr}\big).$$

Démonstration. On a

$$\psi_{pq}(x\otimes y) = \begin{cases} xy & \text{si } \mathrm{d}^\circ(x)=0 \\ x\wedge dy & \text{si } \mathrm{d}^\circ(x)>0 \text{ et } \mathrm{d}^\circ(y)=q \\ 0 & \text{sinon.} \end{cases}$$

Alors

$$\psi_{p+q,r}\big(\psi_{pq}(x\otimes y)\otimes z\big) = \begin{cases} xyz & \text{si } \mathrm{d}^\circ(x)=\mathrm{d}^\circ(y)=0 \\ \psi_{pq}(x\otimes y)\wedge dz & \text{si } \mathrm{d}^\circ(x)+\mathrm{d}^\circ(y)>0 \text{ et } \mathrm{d}^\circ(z)=r \\ 0 & \text{sinon.} \end{cases}$$

Le deuxième cas se décompose en plusieurs sous-cas :

$$\begin{cases} xy\wedge dz & \text{si } \mathrm{d}^\circ(x)=0,\ \mathrm{d}^\circ(y)>0,\ \mathrm{d}^\circ(z)=r \\ x\wedge dy\wedge dz & \text{si } \mathrm{d}^\circ(x)>0,\ \mathrm{d}^\circ(y)=q,\ \mathrm{d}^\circ(z)=r \\ 0 & \text{sinon.} \end{cases}$$

D'autre part,

$$\psi_{p,q+r}\big(x\otimes\psi_{qr}(y\otimes z)\big) = \begin{cases} x\,\psi_{qr}(y\otimes z) & \text{si } \mathrm{d}^\circ(x)=0 \\ x\wedge d\psi_{qr}(y\otimes z) & \text{si } \mathrm{d}^\circ(x)>0,\ \mathrm{d}^\circ(y)=q,\ \mathrm{d}^\circ(z)=r \\ 0 & \text{sinon.} \end{cases}$$

On obtient

$$\begin{cases} xyz & \text{si } \mathrm{d}^\circ(x)=\mathrm{d}^\circ(y)=0 \\ xy\wedge dz & \text{si } \mathrm{d}^\circ(x)=0,\ \mathrm{d}^\circ(y)>0,\ \mathrm{d}^\circ(z)=r \\ x\wedge dy\wedge dz & \text{si } \mathrm{d}^\circ(x)>0,\ \mathrm{d}^\circ(y)=q,\ \mathrm{d}^\circ(z)=r \\ 0 & \text{sinon.} \end{cases}$$

□

Corollaire 3.3. *Si $\alpha\in H^i_{\mathrm{Del}}(X,\mathbb{Z}(p))$, $\beta\in H^j_{\mathrm{Del}}(X,\mathbb{Z}(q))$, $\gamma\in H^k_{\mathrm{Del}}(X,\mathbb{Z}(r))$,*

$$(\alpha\cup\beta)\cup\gamma = \alpha\cup(\beta\cup\gamma).$$

Nous allons maintenant étendre le produit aux complexes $\mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant p} \longrightarrow \Omega_X^{\bullet}\big)[-1]$ de manière compatible. On définit

$$\widetilde{\psi}_{pq} : \mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant p} \to \Omega_X^{\bullet}\big)[-1] \otimes_{\mathbb{Z}} \mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant q} \to \Omega_X^{\bullet}\big)[-1] \longrightarrow \mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant p+q} \to \Omega_X^{\bullet}\big)[-1]$$

de la façon suivante :

$$\widetilde{\psi}_{pq}\big[(\beta,\alpha)\otimes(\beta',\alpha')\big] = \big((-1)^{\mathrm{d}^\circ(\alpha')}\beta\wedge\alpha', \alpha\wedge\alpha'\big) \tag{1}$$

$$\widetilde{\psi}_{pq}\big[(0,n)\otimes(\beta',\alpha')\big] = (n\beta',0) \tag{2}$$

$$\widetilde{\psi}_{pq}\big[(\beta,\alpha)\otimes(0,n)\big] = (0,0) \tag{3}$$

$$\widetilde{\psi}_{pq}\big[(0,n)\otimes(0,n')\big] = (0,nn'). \tag{4}$$

Lemme 3.4. $\widetilde{\psi}_{pq}$ *est un morphisme de complexes.*

Démonstration. On envisage successivement les quatre cas précédents.

Cas (1)

$$\begin{aligned}
&\widetilde{\psi}_{pq}\Big[\delta(\beta,\alpha)\otimes(\beta',\alpha') + (-1)^{\mathrm{d}^\circ(\alpha)}(\beta,\alpha)\otimes\delta(\beta',\alpha')\Big]\\
&\qquad = \widetilde{\psi}_{pq}\Big[\big(d\beta + (-1)^{\mathrm{d}^\circ(\beta)}\alpha, d\alpha\big)\otimes(\beta',\alpha')\Big] + (-1)^{\mathrm{d}^\circ(\alpha)}\widetilde{\psi}_{pq}\Big[(\beta,\alpha)\otimes(?,d\alpha')\Big]\\
&\qquad = \Big((-1)^{\mathrm{d}^\circ(\alpha')}\big(d\beta + (-1)^{\mathrm{d}^\circ(\beta)}\alpha\big)\wedge\alpha' + (-1)^{\mathrm{d}^\circ(\alpha)+\mathrm{d}^\circ(\alpha')+1}\beta\wedge d\alpha', d\alpha\wedge\alpha'\\
&\qquad\qquad + (-1)^{\mathrm{d}^\circ(\alpha)}\alpha\wedge d\alpha'\Big)\\
&\qquad = \Big((-1)^{\mathrm{d}^\circ(\alpha')}d(\beta\wedge\alpha') + (-1)^{\mathrm{d}^\circ(\alpha')+\mathrm{d}^\circ(\beta)}\alpha\wedge\alpha', d(\alpha\wedge\alpha')\Big)\\
&\qquad = \delta\big((-1)^{\mathrm{d}^\circ(\alpha')}\beta\wedge\alpha', \alpha\wedge\alpha'\big) = \delta\,\widetilde{\psi}_{pq}\Big[(\beta,\alpha)\otimes(\beta',\alpha')\Big].
\end{aligned}$$

Cas (2)

$$\begin{aligned}
&\widetilde{\psi}_{pq}\Big[\delta(0,n)\otimes(\beta',\alpha') + (0,n)\otimes\delta(\beta',\alpha')\Big]\\
&\qquad = \widetilde{\psi}_{pq}\Big[(n,0)\otimes(\beta',\alpha')\Big] + \widetilde{\psi}_{pq}\Big[(0,n)\otimes\big(d\beta' + (-1)^{\mathrm{d}^\circ(\beta')}\alpha', d\alpha'\big)\Big]\\
&\qquad = \big((-1)^{\mathrm{d}^\circ(\alpha')}n\alpha', 0\big) + \big(n\,d\beta' + (-1)^{\mathrm{d}^\circ(\beta')}n\alpha', 0\big)\\
&\qquad = (n\,d\beta',0) = \delta(n\beta',0) = \delta\,\widetilde{\psi}_{pq}\Big[(0,n)\otimes(\beta',\alpha')\Big].
\end{aligned}$$

Cas (3)

$$\begin{aligned}
&\widetilde{\psi}_{pq}\Big[\delta(\beta,\alpha)\otimes(0,n') + (-1)^{\mathrm{d}^\circ(\alpha)}(\beta,\alpha)\otimes\delta(0,n')\Big]\\
&\qquad = \widetilde{\psi}_{pq}\Big[\big(d\beta + (-1)^{\mathrm{d}^\circ(\beta)}\alpha, d\alpha\big)\otimes(0,n')\Big] + (-1)^{\mathrm{d}^\circ(\alpha)}\widetilde{\psi}_{pq}\Big[(\beta,\alpha)\otimes(n',0)\Big]\\
&\qquad = 0 = \delta\,\widetilde{\psi}_{pq}\Big[(\beta,\alpha)\otimes(0,n')\Big].
\end{aligned}$$

Cas (4)

$$\begin{aligned}
&\widetilde{\psi}_{pq}\Big[\delta(0,n)\otimes(0,n') + (0,n)\otimes\delta(0,n')\Big]\\
&\qquad \widetilde{\psi}_{pq}\Big[(0,n)\otimes(n',0)\Big] = (nn',0) = \delta\,\widetilde{\psi}_{pq}\Big[(0,n)\otimes(0,n')\Big].
\end{aligned}$$

□

Lemme 3.5. *Le diagramme*

$$\begin{tikzcd}
\mathbb{Z}'_{D,X}(p) \otimes_{\mathbb{Z}} \mathbb{Z}'_{D,X}(q) \arrow[r, "u_p\otimes u_q"] \arrow[d, "\psi_{pq}"'] & \mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet\geqslant p} \longrightarrow \Omega_X^\bullet\big)[-1] \otimes_{\mathbb{Z}} \mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet\geqslant q} \longrightarrow \Omega_X^\bullet\big)[-1] \arrow[d, "\widetilde{\psi}_{pq}"] \\
\mathbb{Z}'_{D,X}(p+q) \arrow[r, "u_{p+q}"] & \mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet\geqslant p+q} \longrightarrow \Omega_X^\bullet\big)[-1]
\end{tikzcd}$$

commute.

Démonstration. Soit x un élément de $\mathbb{Z}'_{D,X}(p)$ et y un élément de $\mathbb{Z}'_{D,X}(q)$. On distingue plusieurs cas :

1. Supposons que $\mathrm{d}^\circ(x) = 0$.

(i) Si $\mathrm{d}^\circ(y) > 0$, $u_p(x) = (0, x)$ et $u_q(y) = (y, ?)$. D'où

$$\widetilde{\psi}_{pq}\big(u_p(x) \otimes u_q(y)\big) = (xy, 0) = u_{p+q}\big(\psi_{pq}(x \otimes y)\big).$$

(ii) Si $\mathrm{d}^\circ(y) = 0$, $u_q(y) = (0, y)$ et

$$\widetilde{\psi}_{pq}\big(u_p(x) \otimes u_q(y)\big) = (0, xy) = u_{p+q}\big(\psi_{pq}(x \otimes y)\big).$$

2. Supposons que $\mathrm{d}^\circ(x) > 0$.

(i) Si $\mathrm{d}^\circ(y) = 0$, $u_q(y) = (0, y)$. Donc

$$\widetilde{\psi}_{pq}\big(u_p(x) \otimes u_q(y)\big) = 0 = u_{p+q}\big(\psi_{pq}(x \otimes y)\big).$$

(ii) Si $\mathrm{d}^\circ(y) < q$, $u_q(y) = (y, 0)$. Donc

$$\widetilde{\psi}_{pq}\big(u_p(x) \otimes u_q(y)\big) = 0 = u_{p+q}\big(\psi_{pq}(x \otimes y)\big).$$

(iii) Si $\mathrm{d}^\circ(y) = q$, $u_q(y) = \big(y, (-1)^q dy\big)$.

(α) Si $\mathrm{d}^\circ(x) < p$, $u_p(x) = (x, 0)$ et

$$\widetilde{\psi}_{pq}\big(u_p(x) \otimes u_q(y)\big) = (x \wedge dy, 0) = u_{p+q}\big(\psi_{pq}(x \otimes y)\big).$$

(β) Si $\mathrm{d}^\circ(x) = p$, $u_p(x) = \big(x, (-1)^p dx\big)$ et donc

$$\widetilde{\psi}_{pq}\big(u_p(x) \otimes u_q(y)\big) = \big(x \wedge dy, (-1)^{p+q} dx \wedge dy)\big) = \big(x \wedge dy, (-1)^{p+q} d(x \wedge dy)\big) = u_{p+q}\big(\psi_{pq}(x \otimes y)\big).$$

$\square$

Ceci montre que $\widetilde{\psi}_{pq}$ induit en cohomologie le cup-produit défini par ψ_{pq} via les isomorphismes u_p et u_q.

Nous aurons besoin dans la suite des cup-produits « mixtes » au niveau des complexes, ce qui signifie que l'on utilise le complexe de Deligne usuel pour une variable et le complexe de Deligne donné par un cône pour l'autre variable. Soit

$$\phi_{pq} : \mathbb{Z}'_{D,X}(p) \otimes_{\mathbb{Z}} \mathrm{Mc}\big(\mathbb{Z} \oplus \Omega_X^{\bullet\geqslant q} \longrightarrow \Omega_X^\bullet\big)[-1] \xrightarrow[\widetilde{\psi}_{pq}(u_p\otimes \mathrm{id})]{} \mathrm{Mc}\big(\mathbb{Z} \oplus \Omega_X^{\bullet\geqslant p+q} \longrightarrow \Omega_X^\bullet\big)[-1].$$

On a alors

$$\phi_{pq}\big(x\otimes(\beta',\alpha')\big)=\begin{cases}(x\beta',0) & \text{si } \mathrm{d}^\circ(x)=0\\ \big((-1)^{\mathrm{d}^\circ(\alpha')}x\wedge\alpha',0\big) & \text{si } 0<\mathrm{d}^\circ(x)<p\\ \big((-1)^{\mathrm{d}^\circ(\alpha')}x\wedge\alpha',(-1)^p dx\wedge\alpha'\big) & \text{si } \mathrm{d}^\circ(x)=p.\end{cases}$$

et

$$\phi_{pq}\big(x\otimes(0,n')\big)=\begin{cases}(0,xn') & \text{si } \mathrm{d}^\circ(x)=0\\ 0 & \text{si } \mathrm{d}^\circ(x)>0.\end{cases}$$

On a également dans l'autre sens le produit

$$\Gamma_{pq}:\mathrm{Mc}\big(\mathbb{Z}\oplus\Omega_X^{\bullet\geqslant p}\longrightarrow\Omega_X^\bullet\big)[-1]\otimes_{\mathbb{Z}}\mathbb{Z}'_{D,X}(q)\xrightarrow[\widetilde{\psi}_{pq}(\mathrm{id}\otimes u_q)]{}\mathrm{Mc}\big(\mathbb{Z}\oplus\Omega_X^{\bullet\geqslant p+q}\longrightarrow\Omega_X^\bullet\big)[-1].$$

donné par :

$$\Gamma_{pq}\big((\beta,\alpha)\otimes y\big)=\begin{cases}0 & \text{si } \mathrm{d}^\circ(y)<q\\ \big(\beta\wedge dy,(-1)^q\alpha\wedge dy\big) & \text{si } \mathrm{d}^\circ(y)=q\end{cases}$$

et

$$\Gamma_{pq}\big((0,n)\otimes y\big)=\begin{cases}(ny,0) & \text{si } \mathrm{d}^\circ(y)>0\\ (0,ny) & \text{si } \mathrm{d}^\circ(y)=0.\end{cases}$$

On admet provisoirement le résultat suivant, qui sera établi dans la section 6 :

Proposition 3.6. *Soit $\mathcal{D}_X^\bullet$ le complexe de de Rham des courants sur X. Il existe alors un sous-complexe $\mathcal{D}_{X,\mathbb{Z}}^\bullet$ de $\mathcal{D}_X^\bullet$ quasi-isomorphe à $\mathbb{Z}_X$ formé de faisceaux mous.*

On en déduit que $\mathbb{Z}'_{D,X}(p)$ est quasi-isomorphe au complexe $\mathrm{Mc}\big(\mathcal{D}_{X,\mathbb{Z}}^\bullet\oplus F^p\mathcal{D}_X^\bullet\longrightarrow\mathcal{D}_X^\bullet\big)[-1]$ et que la cohomologie des sections globales de ce complexe en degré k est $H^k_{\mathrm{Del}}\big(X,\mathbb{Z}(p)\big)$. Une classe de cohomologie dans $H^k_{\mathrm{Del}}\big(X,\mathbb{Z}(p)\big)$ est donc représentée par un triplet $(S,T\oplus U)$ tel que $S\in\mathcal{D}_X^{k-1}$, $T\in\mathcal{D}_{X,\mathbb{Z}}^k$, $U\in F^p\mathcal{D}_X^k$ et

$$dS+(-1)^{k-1}(-T+U)=0,\quad dT=dU=0.$$

On définit maintenant

$$\Delta:\mathrm{Mc}\big(\mathcal{D}_{X,\mathbb{Z}}^\bullet\oplus F^p\mathcal{D}_X^\bullet\longrightarrow\mathcal{D}_X^\bullet\big)[-1]\otimes_{\mathbb{Z}}\mathbb{Z}'_{D,X}(q)\longrightarrow\mathrm{Mc}\big(\mathcal{D}_{X,\mathbb{Z}}^\bullet\oplus F^{p+q}\mathcal{D}_X^\bullet\longrightarrow\mathcal{D}_X^\bullet\big)[-1]$$

en posant

$$\Delta\big[(S,T\oplus U),y\big]=\begin{cases}(0,yT) & \text{si } \mathrm{d}^\circ(y)=0\\ (T\wedge y,0) & \text{si } 0<\mathrm{d}^\circ(y)<q\\ \big(T\wedge y+S\wedge dy,(-1)^q\,U\wedge dy\big) & \text{si } \mathrm{d}^\circ(y)=q.\end{cases}$$

Lemme 3.7. *Δ est un morphisme de complexes.*

Démonstration. On doit montrer

$$\Delta\big[\delta(S,T\oplus U)\otimes y\big]+(-1)^{d^\circ T}\Delta\big[(S,T\oplus U)\otimes\delta y\big]=\delta\,\Delta\big[(S,T\oplus U)\otimes y\big].$$

Soit $(S,T\oplus U)$ un élément de degré k. On note $(S',T'\oplus U')=\delta(S,T\oplus U)$. On a

$$S'=dS+(-1)^k(T-U),\quad T'=dT,\quad U'=dU.$$

On considère les différents cas :

1. $\mathrm{d}^\circ(y) = 0$. Alors

$$\begin{aligned}\Delta\big[(S', T' \oplus U') \otimes y\big] &= (0, yT') = \big((-1)^k yT, ydT\big) + (-1)^{k-1}(yT, 0)\\ &= \delta\,\Delta\big[(S, T \oplus U) \otimes y\big] + (-1)^{k-1}\Delta\big[(S, T \oplus U) \otimes \delta y\big].\end{aligned}$$

2. $0 < \mathrm{d}^\circ(y) < q$. Alors

$$\begin{aligned}\Delta\big[(S', T' \oplus U') \otimes y\big] &= (T' \wedge y, 0) = \big(d(T \wedge y), 0\big) + (-1)^{k-1}(T \wedge dy, 0)\\ &= \delta\,\Delta\big[(S, T \oplus U) \otimes y\big] + (-1)^{k-1}\Delta\big[(S, T \oplus U) \otimes \delta y\big].\end{aligned}$$

3. $\mathrm{d}^\circ(y) = q$. Alors

$$\begin{aligned}\Delta\big[(S', T' \oplus U') \otimes y\big] &= \big(T' \wedge y + S' \wedge dy, (-1)^q\, U' \wedge dy\big)\\ &= \Big(dT \wedge y + \big(dS + (-1)^k(T - U)\big) \wedge dy, (-1)^q dU \wedge dy\Big)\\ &= \Big(d(T \wedge y + S \wedge dy) + (-1)^{k+q-1}\big[(-1)^q\, U \wedge dy\big], (-1)^q d(U \wedge dy)\Big)\\ &= \delta\,\Delta\big[(S, T \oplus U) \otimes y\big].\end{aligned}$$

□

LEMME 3.8. *Le diagramme suivant commute :*

$$\begin{tikzcd}
\mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant p} \longrightarrow \Omega_X^\bullet\big)[-1] \otimes_{\mathbb{Z}} \mathbb{Z}'_{D,X}(q) \arrow[r, "\simeq"] \arrow[d, "\Gamma_{pq}"] & \mathrm{Mc}\big(\mathcal{D}_{X,\mathbb{Z}}^\bullet \oplus F^p\mathcal{D}_X^\bullet \longrightarrow \mathcal{D}_X^\bullet\big)[-1] \otimes_{\mathbb{Z}} \mathbb{Z}'_{D,X}(q) \arrow[d, "\Delta"] \\
\mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant p+q} \longrightarrow \Omega_X^\bullet\big)[-1] \arrow[r, "\simeq"'] & \mathrm{Mc}\big(\mathcal{D}_{X,\mathbb{Z}}^\bullet \oplus F^{p+q}\mathcal{D}_X^\bullet \longrightarrow \mathcal{D}_X^\bullet\big)[-1]
\end{tikzcd}$$

DÉMONSTRATION. On distingue les cas :

1. Si $\mathrm{d}^\circ(y) < q$,

(i) $\Gamma_{pq}\big((\beta, \alpha) \otimes y\big) = 0$ et $\Delta\big((\beta, 0 \oplus \alpha) \otimes y\big) = 0$.

(ii)

$$\Gamma_{pq}\big((0, n) \otimes y\big) = \begin{cases}(ny, 0)\\(0, ny)\end{cases} \qquad \text{et} \qquad \Delta\big((0, n \oplus 0) \otimes y\big) = \begin{cases}(ny, 0)\\(0, ny)\end{cases}$$

selon que $\mathrm{d}^\circ(y)$ est strictement positif ou bien nul.

2. Si $\mathrm{d}^\circ(y) = q$,

(i) $\Gamma_{pq}\big((\beta, \alpha) \otimes y\big) = \big(\beta \wedge dy, (-1)^q \alpha \wedge dy\big) = \Delta\big((\beta, 0 \oplus \alpha) \otimes y\big)$

(ii) $\Gamma_{pq}\big((0, n) \otimes y\big) = (ny, 0) = \Delta\big((0, n \oplus 0) \otimes y\big)$.

COROLLAIRE 3.9. Δ *induit le cup-produit en cohomologie.*

□

4. Classe de Bloch

Soit X une variété analytique complexe lisse et Z inclus dans X un cycle localement intersection complète de codimension d.

PROPOSITION 4.1. *La cohomologie locale de $\mathcal{O}_X$ à support dans Z est concentrée en degré d.*

DÉMONSTRATION. Comme on raisonne localement, on peut supposer que X est une variété de Stein. Si $d = 1$, $X \setminus Z$ est aussi une variété de Stein et le lemme est conséquence de Cartan B. Montrons maintenant le cas général. On écrit $Z = \{f_1 = 0\} \cap \cdots \cap \{f_d = 0\}$. Soit $U = X \setminus Z$ et U_i l'ensemble des x de X tels que $f_i(x) \neq 0$. Les U_i forment un recouvrement de Stein de U. La cohomologie de $\mathcal{O}_X$ peut donc se calculer en prenant la cohomologie du complexe de Čech associé aux U_i (par Cartan B). Ce complexe est concentré en degré au plus $d - 1$, donc $H^i\big(U, \mathcal{O}_X\big) = 0$ pour $i \geq d$. Par suite $H^i_Z\big(X, \mathcal{O}_X\big) = 0$ pour $i > d$.

On utilise maintenant le résultat suivant [**Sche**] : Soit X une variété compacte et A un sous-ensemble analytique de codimension au moins r. Alors pour tout faisceau localement libre $\mathcal{S}$ sur X et tout entier q tel que $q \leq r - 2$,

$$H^q(X, \mathcal{S}) \xrightarrow{\simeq} H^q(X \setminus A, \mathcal{S}).$$

Ceci montre que $H^i\big(U, \mathcal{O}_X\big) = 0$ pour $i \leq d - 2$, d'où $H^i_Z\big(X, \mathcal{O}_X\big) = 0$ pour $i \leq d - 1$. $\square$

On a un résultat analogue pour la cohomologie locale algébrique à support dans Z. Rappelons que

$$\mathcal{R}\Gamma_{[Z]}\big(\mathcal{O}_X\big) = \varinjlim_j \mathcal{R}\mathcal{H}om_{\mathcal{O}_X}\big(\mathcal{O}_X/_{\mathcal{J}^j_Z}, \mathcal{O}_X\big).$$

Alors

PROPOSITION 4.2. *Le complexe $\mathcal{R}\Gamma_{[Z]}(\mathcal{O}_X)$ est concentré localement en degré d.*

DÉMONSTRATION. – Écrivons $Z = Z_1 \cap \cdots \cap Z_d$ où les Z_i sont des hypersurfaces. Alors

$$\mathcal{R}\Gamma_{[Z]}\big(\mathcal{O}_X\big) = \mathcal{R}\Gamma_{[Z_1]} \ldots \mathcal{R}\Gamma_{[Z_d]}\big(\mathcal{O}_X\big).$$

Si $\mathcal{F}$ est un $\mathcal{O}_X$-module (non nécessairement cohérent) et $W = \{f = 0\}$ une hypersurface, alors $\mathcal{R}\Gamma_{[Z]}(\mathcal{F})$ est concentré en degrés au plus 1 car $\mathcal{O}_X/_{\mathcal{J}^j_W}$ est quasi-isomorphe à $\mathcal{O}_X \xrightarrow{f^j} \mathcal{O}_X$. De cela on déduit que $\mathcal{R}\Gamma_{[Z]}\big(\mathcal{O}_X\big)$ est concentré en degrés au plus d.

– On a $\mathcal{E}xt^j_{\mathcal{O}_X}\big(\mathcal{O}_X/_{\mathcal{J}^j_Z}, \mathcal{O}_X\big) = 0$ pour $j < \operatorname{codim}\big(\operatorname{supp}(\mathcal{O}_X/_{\mathcal{J}^j_Z})\big) = d$, donc $\mathcal{R}\Gamma_{[Z]}\big(\mathcal{O}_X\big)$ est concentré en degrés au moins d. $\square$

NOTATION 4.3. *On pose $B_{Z|X} = \mathcal{H}^d\big(\mathcal{R}\Gamma_{[Z]}\big(\mathcal{O}_X\big)\big)$ et $B^{\infty}_{Z|X} = \mathcal{H}^d\big(\mathcal{R}\Gamma_Z\big(\mathcal{O}_X\big)\big)$.*

On a donc $\mathcal{R}\Gamma_{[Z]}\big(\mathcal{O}_X\big) \simeq B_{Z|X}[-d]$ et $\mathcal{R}\Gamma_Z\big(\mathcal{O}_X\big) \simeq B^{\infty}_{Z|X}[-d]$. Les préfaisceaux

$$U \longrightarrow \varinjlim_j \operatorname{Ext}^d_{\mathcal{O}_X}\big(U, \mathcal{O}_X/_{\mathcal{J}^j_Z}, \mathcal{O}_X\big) \quad \text{et} \quad U \longrightarrow H^d_Z\big(U, \mathcal{O}_X\big)$$

sont des faisceaux, ce sont $B_{Z|X}$ et $B^{\infty}_{Z|X}$. Les faisceaux $B_{Z|X}$ et $B^{\infty}_{Z|X}$ ne sont pas de type fini sur $\mathcal{O}_X$. Ce sont des $\mathcal{D}_X$-modules, $B_{Z|X}$ est cohérent sur $\mathcal{D}_X$, contrairement à $B^{\infty}_{Z|X}$ qui n'est même pas de type fini sur $\mathcal{D}_X$ ([**Bj**] section 2.5 et [**Ka**] section 3.4).

PROPOSITION 4.4. $\mathcal{R}\Gamma_Z\big(\Omega_X^{\bullet\geqslant p}\big)$ *est quasi-isomorphe à*

$$\big(\ B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^p_X \longrightarrow B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^{p+1}_X \longrightarrow \cdots \longrightarrow B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^n_X\ \big)[-d],$$

les différentielles étant induites par d. Dans cette écriture, Ω^p_X est en degré p.

DÉMONSTRATION. On présente deux preuves de ce résultat.

1. La première preuve utilise la théorie des $\mathcal{D}_X$-modules et m'a été communiquée par P. Schapira. Rappelons que le complexe de Spencer de $\mathcal{D}_X$, noté $\mathrm{Sp}\big(\mathcal{D}_X\big)$, est donné par

$$\mathcal{D}_X\otimes_{\mathcal{O}_X}\bigwedge^n TX \longrightarrow \cdots \longrightarrow \mathcal{D}_X\otimes_{\mathcal{O}_X}\bigwedge^1 TX \longrightarrow \mathcal{D}_X$$

où $\mathcal{D}_X$ est en degré 0 et les flèches sont définies par

$$\delta\big[P\otimes v_1\wedge\ldots\wedge v_k\big]=\sum_{j=1}^k(-1)^{j-1}(Pv_j)v_1\wedge\ldots v_{j-1}\wedge v_{j+1}\wedge\ldots\wedge v_k.$$

Le complexe $\mathrm{Sp}\big(\mathcal{D}_X\big)$ est en coordonnées locales le complexe de Koszul de $\mathcal{D}_X$ associé à la suite régulière $\partial/_{\partial z_1},\ldots,\partial/_{\partial z_n}$, il est donc quasi-isomorphe à sa cohomologie en degré 0 qui est

$$\mathcal{D}_X\Big/_{\mathcal{D}_X\partial/_{\partial z_1}+\cdots+\mathcal{D}_X\partial/_{\partial z_n}}=\mathcal{O}_X.$$

Le complexe $\mathrm{Sp}\big(\mathcal{D}_X\big)$ est ainsi une résolution $\mathcal{D}_X$-libre de $\mathcal{O}_X$.
Le complexe $\mathrm{Sp}\big(\mathcal{D}_X\big)^{\leqslant -p}$ est obtenu en supprimant les p derniers termes. Alors

$$\mathcal{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathrm{Sp}\big(\mathcal{D}_X\big)^{\leqslant -p},\mathcal{O}_X\big)\simeq\Omega_X^{\bullet\geqslant p}.$$

On en déduit

$$\begin{aligned}\mathcal{R}\Gamma_Z\Omega_X^{\bullet\geqslant p}&\simeq\mathcal{R}\Gamma_Z\,\mathcal{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathrm{Sp}\big(\mathcal{D}_X\big)^{\leqslant -p},\mathcal{O}_X\big)\\&\simeq\mathcal{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathrm{Sp}\big(\mathcal{D}_X\big)^{\leqslant -p},\mathcal{R}\Gamma_Z\,\mathcal{O}_X\big)\\&\simeq\mathcal{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathrm{Sp}\big(\mathcal{D}_X\big)^{\leqslant -p},B^\infty_{Z|X}[-d]\big)\\&\simeq\mathcal{R}\mathcal{H}om_{\mathcal{D}_X}\big(\mathrm{Sp}\big(\mathcal{D}_X\big)^{\leqslant -p},B^\infty_{Z|X}\big)[-d].\end{aligned}$$

2. La deuxième preuve se base sur le lemme suivant :

LEMME 4.5. *Soient $\mathcal{C}$ et $\mathcal{C}'$ deux catégories abéliennes admettant suffisamment d'injectifs et F un foncteur exact à gauche de $\mathcal{C}$ dans $\mathcal{C}'$. Soit $A^\bullet$ un complexe borné de $\mathcal{C}$ et d un entier tel que $\forall i\neq d$, $\forall j$, $R^iF(A^j)=0$. Alors $\mathbb{R}\,F(A^\bullet)\simeq R^dF(A^\bullet)[-d]$.*

DÉMONSTRATION. La preuve se fait par récurrence sur la longueur de $A^\bullet$. On peut supposer que A est concentré en degrés positifs. Si $A^\bullet$ est concentré en degré zéro, $\mathbb{R}F(A^\bullet)$ est concentré en degré d et est donc quasi-isomorphe à sa cohomologie en degré d, à savoir $R^dF(A^\bullet)$. Si $A^\bullet$ est concentré en degrés plus petits que j, on considère le diagramme de triangles distingués

$$\begin{tikzcd}\mathbb{R}F(A^j)[-j] \arrow[r] \arrow[d, "\simeq"'] & \mathbb{R}F\big(A^\bullet\big) \arrow[r] & \mathbb{R}F\big(A^{\bullet\leq j-1}\big) \arrow[r, "+1"] \arrow[d, "\simeq"'] & {} \\ R^dF(A^j)[-d-j] \arrow[r] & R^dF(A^\bullet)[-d] \arrow[r] & R^dF\big(A^{\bullet\leq j-1}\big) \arrow[r, "+1"] & {}\end{tikzcd}$$

où le dernier quasi-isomorphisme vertical est construit par récurrence. Il existe donc un morphisme $\mathbb{R}F(A^\bullet) \longrightarrow R^dF(A^\bullet)[-d]$ qui est par suite un quasi-isomorphisme. $\square$

Le résultat de la proposition 4.4 est alors conséquence du lemme 4.5 et de la proposition 4.1. $\square$

Nous allons maintenant passer à la construction de la classe de Bloch proprement dite. On reprend l'expression

$$B_{Z|X} = \varinjlim_j \mathcal{E}xt^d_{\mathcal{O}_X}\Big(\mathcal{O}_X/_{\mathcal{J}_Z^j}, \mathcal{O}_X\Big).$$

Le théorème de dualité locale (voir [**Gri-Ha**]) fournit un isomorphisme

$$\mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X) \otimes_{\mathcal{O}_Z} \bigwedge^d N^*_{Z/X} \simeq \mathcal{O}_Z.$$

On considère la chaîne de morphismes

$$\mathcal{O}_Z \longrightarrow \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X) \otimes_{\mathcal{O}_Z} \bigwedge^d N^*_{Z/X} \longrightarrow \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X) \otimes_{\mathcal{O}_Z} \bigwedge^d \Omega^1_{X|Z}$$

$$\longrightarrow \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \bigwedge^d \Omega^1_X \longrightarrow \mathcal{E}xt^d_{\mathcal{O}_X}(\mathcal{O}_Z, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \Omega^d_X \longrightarrow B_{Z|X} \otimes_{\mathcal{O}_X} \Omega^d_X.$$

Ce morphisme définit une section globale de $B_{Z|X} \otimes_{\mathcal{O}_X} \Omega^d_X$ appelée la *classe fondamentale* de Z.

LEMME 4.6. *Soient Z et W deux sous-variétés transverses de codimensions d et d' dans X. Alors le morphisme de cup-produit à support $\mathcal{H}^d_Z(\Omega^d_X) \otimes_{\mathcal{O}_X} \mathcal{H}^{d'}_W(\Omega^{d'}_X) \longrightarrow \mathcal{H}^{d+d'}_{Z\cap W}(\Omega^{d+d'}_X)$ envoie les classes fondamentales de Z et W sur la classe fondamentale de $Z \cap W$.*

DÉMONSTRATION. On écrit

$$Z = \{f_1 = 0\} \cap \cdots \cap \{f_d = 0\} \quad \text{et} \quad W = \{g_1 = 0\} \cap \cdots \cap \{g_{d'} = 0\}\cdot$$

Alors

$$B_{Z|X} \simeq B_{\{f_1=0\}|X} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} B_{\{f_d=0\}|X}$$
$$B_{W|X} \simeq B_{\{g_1=0\}|X} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} B_{\{g_{d'}=0\}|X}\cdot$$

Via ces isomorphismes, les classes fondamentales de Z et W sont données par

$$\frac{1}{f_1} \otimes \cdots \otimes \frac{1}{f_d} \otimes df_1 \wedge \cdots \wedge df_d \quad \text{et} \quad \frac{1}{g_1} \otimes \cdots \otimes \frac{1}{g_{d'}} \otimes dg_1 \wedge \cdots \wedge dg_{d'}$$

(voir [**Ka**], section 3.4).
Le cup-product à support $B_{Z|X} \otimes_{\mathcal{O}_X} B_{W|X} \longrightarrow B_{Z\cap W|X}$ est un isomorphisme donné par les isomorphismes

$$\begin{aligned} B_{Z|X} \otimes_{\mathcal{O}_X} B_{W|X} &\simeq \big(B_{\{f_1=0\}|X} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} B_{\{f_d=0\}|X}\big) \otimes_{\mathcal{O}_X} \big(B_{\{g_1=0\}|X} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} B_{\{g_{d'}=0\}|X}\big) \\ &\simeq B_{Z\cap W|X} \end{aligned}$$

Par consequent, le produit des classes fondamentales de Z et W est

$$\frac{1}{f_1} \otimes \cdots \otimes \frac{1}{f_d} \otimes \frac{1}{g_1} \otimes \cdots \otimes \frac{1}{g_{d'}} \otimes df_1 \wedge \cdots \wedge df_d \wedge dg_1 \wedge \cdots \wedge dg_{d'}$$

qui est la classe de cycle de $Z \cap W$. $\square$

LEMME 4.7. *La classe fondamentale s'envoie sur* 0 *via le morphisme*

$$B_{Z|X} \otimes_{\mathcal{O}_X} \Omega^d_X \longrightarrow B_{Z|X} \otimes_{\mathcal{O}_X} \Omega^{d+1}_X.$$

DÉMONSTRATION. Écrivons $Z = Z_1 \cap \cdots \cap Z_d$. On a $B_{Z|X} \simeq B_{Z_1|X} \otimes_{\mathcal{O}_X} \cdots \otimes_{\mathcal{O}_X} B_{Z_d|X}$. Via cet isomorphisme, la classe fondamentale de Z est

$$\left(\frac{1}{f_1} \otimes \cdots \otimes \frac{1}{f_d}\right) \otimes df_1 \wedge \cdots \wedge df_d.$$

Elle est donc fermée. □

On dispose donc d'un morphisme

$$\mathcal{O}_Z \longrightarrow \Big(B_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^d \longrightarrow \cdots\cdots \longrightarrow B_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^n\Big)[d]$$

où Ω_X^d est en degré d, et donc une classe de cohomologie dans

$$\mathbb{H}^d\Big(X,\ B_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^d \longrightarrow \cdots\cdots \longrightarrow B_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^n\Big)$$

qui est la *classe de Bloch algébrique* de Z. En composant avec les morphismes canoniques $\mathbb{Z}_Z \longrightarrow \mathcal{O}_Z$ et $B_{Z|X} \longrightarrow B_{Z|X}^{\infty}$, on obtient par la proposition 4.4 un morphisme

$$\mathbb{Z}_Z[-2d] \longrightarrow \mathcal{R}\,\Gamma_Z\big(X, \Omega_X^{\bullet \geqslant d}\big)$$

qui est le *morphisme de Bloch*, ainsi qu'une classe $\{Z\}_{\mathrm{Bl}}$ dans $\mathcal{H}^{2d}\big(\mathcal{R}\Gamma_Z(\Omega_X^{\bullet\geqslant d})\big) = \mathbb{H}_Z^{2d}\big(X, \Omega_X^{\bullet\geqslant d}\big)$ qui est par définition la *classe de Bloch analytique* de Z.

LEMME 4.8. *Si Z est lisse, l'image de $\{Z\}_{Bl}$ dans $\mathbb{H}_Z^{2d}\big(X, \Omega_X^{\bullet}\big) = H_Z^{2d}(X, \mathbb{C})$ est $(2i\pi)^d\{Z\}_{\mathrm{top}}$.*

DÉMONSTRATION. Il suffit de raisonner localement car $U \longrightarrow H_Z^{2d}(U, \mathbb{C})$ est un faisceau. Si $Z = Z_1 \cap \cdots \cap Z_d$, on a localement $\{Z\}_{\mathrm{Bl}} = \{Z_1\}_{\mathrm{Bl}} \cdots \{Z_d\}_{\mathrm{Bl}}$ et $\{Z\}_{\mathrm{top}} = \{Z_1\}_{\mathrm{top}} \ldots \{Z_d\}_{\mathrm{top}}$. On peut donc supposer que Z est une hypersurface. La théorie de Leray fournit un diagramme commutatif :

$$\begin{tikzcd}
H^1(U, \mathbb{C}) \arrow[r, "\delta"] \arrow[d, "\simeq"'] & H_Z^2(X, \mathbb{C}) \\
\mathbb{H}^1\big(\mathcal{A}_X^{\bullet}(\log Z)\big) \arrow[r, "2i\pi\,\mathrm{res}"'] & H^0(Z, \mathbb{C}) \arrow[u, "i_{Z*}"']
\end{tikzcd}$$

où $U = X \setminus Z$. D'autre part, on a un diagramme

$$\begin{tikzcd}
\mathbb{H}^1\big(U, \Omega_X^{\bullet\geqslant 1}\big) \arrow[r, "\delta"] \arrow[d] & \mathbb{H}_Z^2(X, \Omega_X^{\bullet\geqslant 1}) \arrow[d] \\
H^1(U, \mathbb{C}) \arrow[r, "\delta"'] & H_Z^2(X, \mathbb{C})
\end{tikzcd}$$

Comme $\mathrm{Res}\big(df/f\big) = 1$, on obtient le résultat. □

5. La classe de cycle en cohomologie de Deligne

Rappelons que

$$\mathbb{Z}_{D,X}(p) = \mathrm{Mc}\Big(\mathbb{Z}_X \oplus \Omega_X^{\bullet\geqslant p} \xrightarrow[-(2i\pi)^p, i]{} \Omega_X^{\bullet}\Big)[-1]$$

PROPOSITION 5.1. *Soit Z un cycle localement intersection complète de codimension d. Alors*

$$\mathcal{R}\,\Gamma_Z\big(\mathbb{Z}_{D,X}(d)\big) \simeq \mathbb{Z}_Z[-2d] \oplus \Big(B_{Z|X}^{\infty} \to B_{Z|X}^{\infty} \otimes_{\mathcal{O}_X} \Omega_X^1 \to \cdots \to B_{Z|X}^{\infty} \otimes_{\mathcal{O}_X} \Omega_X^{d-1}\Big)[-d-1].$$

DÉMONSTRATION. Le lemme 4.8 montre que le diagramme suivant est commutatif :

$$\begin{tikzcd} \mathbb{Z}_Z \arrow[r, "\simeq"] \arrow[d, "\mathrm{Bl}"'] & \mathcal{R}\,\Gamma_Z(\mathbb{Z}_X)[2d] \arrow[d, "(2i\pi)^p"] \\ \mathcal{R}\,\Gamma_Z(\Omega_X^{\bullet\geqslant d})[2d] \arrow[r] & \mathcal{R}\,\Gamma_Z(\mathbb{C}_X)[2d] \end{tikzcd}$$

On peut donc écrire

$$\mathcal{R}\,\Gamma_Z\big(\mathbb{Z}_{D,X}(d)\big) = \mathrm{Mc}\Big(\mathbb{Z}_Z[-2d] \oplus \mathcal{R}\,\Gamma_Z\big(\Omega_X^{\bullet\geqslant d}\big) \longrightarrow \mathcal{R}\,\Gamma_Z\big(\Omega_X^{\bullet}\big)\Big)[-1],$$

la flèche $\mathbb{Z}_Z \longrightarrow \mathcal{R}\,\Gamma_Z\big(\Omega_X^{\bullet}\big)[2d]$ étant obtenue par l'opposé du morphisme de Bloch analytique. Le morphisme de complexes dont on prend le cône est donné d'après le lemme 4.4 par

$$\begin{tikzcd} & \mathbb{Z}_Z \oplus B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^d \arrow[r] \arrow[d] & B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d-1} \arrow[r] & B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d} \arrow[r] & B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d+1} \arrow[r] & \cdots \end{tikzcd}$$

où $\mathbb{Z}_X$ est en degré $2d$, et la flèche verticale est donnée par $(-\mathrm{Bl}, \mathrm{id})$.

1. Vérifions que le complexe

$$\mathrm{Mc}\Big(\mathbb{Z}_Z[-2d] \oplus \mathcal{R}\,\Gamma_Z\big(\Omega_X^{\bullet\geqslant d}\big) \longrightarrow \mathcal{R}\,\Gamma_Z\big(\Omega_X^{\bullet}\big)\Big)[-1],$$

est exact en degré supérieur ou égal à $2d+1$. Soit $k \geq d$, et (β, α) un couple tel que β appartienne à $B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^k$ et α appartienne à $B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{k+1}$. On suppose que $\delta(\beta, \alpha) = 0$. Alors $\delta\beta + (-1)^{d+k}\alpha = 0$ et $\delta\alpha = 0$. On en déduit que $(\beta, \alpha) = \delta\big(0, (-1)^{d+k+1}\alpha\big)$.

2. On regarde maintenant ce qui se passe en degré $2d$. Soit $(\beta, n+\alpha)$ tel que $\delta\beta - \big(\alpha - n\{Z\}_{\mathrm{Bl}}\big) = 0$ et $\delta\alpha = 0$. Les éléments exacts sont du type $(\delta\gamma, 0)$ où γ appartient à $B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d-2}$. La cohomologie peut donc s'identifier à $\Big[B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d-1} \big/ \delta\big(B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d-2}\big)\Big] \oplus \mathbb{Z}_Z$ via l'application $(\beta, n+\alpha) \longrightarrow (\beta, n)$. En effet, si (β, n) est donné, il suffit de définir $\alpha = \delta\beta + n\{Z\}_{\mathrm{Bl}}$, et alors $\delta\alpha = 0$ car $\delta\{Z\}_{\mathrm{Bl}} = 0$. Le morphisme naturel

$$\mathbb{Z}_Z[-2d] \oplus \Big(B^{\infty}_{Z|X} \longrightarrow B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^1 \longrightarrow \cdots \longrightarrow B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d-1}\Big)[-d-1] \longrightarrow \mathcal{R}\Gamma_Z\big(\mathbb{Z}_{D,X}(d)\big)$$

est donc un quasi-isomorphisme. □

COROLLAIRE 5.2. *Il existe une classe de cycle canonique* $\{Z\}_D$ *dans* $H_Z^{2d}\big(X, \mathbb{Z}_{D,X}(d)\big)$ *qui est envoyée sur* $\{Z\}_{\mathrm{Bl}}$ *et sur* $(2i\pi)^d\,\{Z\}_{\mathrm{top}}$ *par les morphismes* $H_Z^{2d}\big(X, \mathbb{Z}_{D,X}(d)\big) \longrightarrow H_Z^{2d}\big(X, \Omega_X^{\bullet\geqslant p}\big)$. *et* $H_Z^{2d}\big(X, \mathbb{Z}_{D,X}(d)\big) \longrightarrow H_Z^{2d}\big(X, \mathbb{Z}_X\big)$

REMARQUE 5.3. *Pour obtenir le corollaire 5.2, on peut raisonner uniquement en cohomologie et non au niveau des complexes. Cependant la propriété précédente est plus intéressante car elle permet d'écrire le scindage*

$$\mathcal{R}\Gamma_Z\big(\mathbb{Z}_{D,X}(d)\big) \simeq \mathbb{Z}_Z[-2d] \oplus \mathcal{F}[-2d] \quad \textit{où} \quad \mathcal{F} = B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d-1} \big/ \delta\big(B_{Z|X} \otimes_{\mathcal{O}_X} \Omega_X^{d-2}\big)\,.$$

Cette approche permet également de construire lorsque Z est lisse un morphisme

$$i_{Z*}\colon H^k_{Del}(Z,\mathbb{Z}(p)) \longrightarrow \mathbb{H}^{k+2d}_{Z,D}(X,\mathbb{Z}(p+d))$$

comme expliqué dans le corollaire 5.5.

PROPOSITION 5.4. *Si $p > d$, $\mathrm{R}\Gamma_Z\big(\mathbb{Z}_{D,X}(p)\big)$ est quasi-isomorphe à*

$$\mathrm{Mc}\Big(\mathbb{Z}_Z[-2d] \longrightarrow \Big\{B^\infty_{Z|X} \longrightarrow B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^1_X \longrightarrow \cdots \longrightarrow B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^{p-1}_X\Big\}[-d]\Big)\ [-1],$$

le morphisme $\mathbb{Z}_Z \longrightarrow B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^d_X$ étant l'opposé du morphisme de Bloch.

DÉMONSTRATION. Comme avant, $\mathcal{R}\Gamma_Z\big(\mathbb{Z}_{D,X}(p)\big)$ est quasi-isomorphe au cône du morphisme de complexes suivant :

$$\begin{tikzcd}
 & \mathbb{Z}_Z \arrow[d] & & B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^p_X \arrow[r]\arrow[d] & B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^{p+1}_X \arrow[r]\arrow[d] & \cdots \\
\cdots \arrow[r] & B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^d_X \arrow[r] & \cdots \arrow[r] & B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^p_X \arrow[r] & B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^{p+1}_X \arrow[r] & \cdots
\end{tikzcd}$$

décalé de -1, où $\mathbb{Z}_Z$ est en degré $2d$. La cohomologie est ainsi concentrée en degrés au plus $d+p-1$. On sait que $\mathcal{R}\Gamma_Z\big(\Omega^\bullet_X\big) \simeq \mathcal{R}\Gamma_Z\big(\mathbb{C}_X\big) \simeq \mathbb{C}_Z[-2d]$. La cohomologie du complexe $B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^\bullet_X$ se situe donc sur le terme $B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^d_X$. Le complexe $\mathcal{R}\Gamma_Z$ est ainsi quasi-isomorphe au cône du morphisme

$$\begin{tikzcd}
 & \mathbb{Z}_Z \arrow[d] & & \\
\cdots \arrow[r] & B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^d_X \arrow[r] & \cdots \arrow[r] & B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^{p-1}_X
\end{tikzcd}$$

□

COROLLAIRE 5.5. *Supposons Z lisse. On dispose alors d'un morphisme canonique*

$$i_{Z*}\colon \mathbb{Z}_{D,X}(p) \longrightarrow \mathcal{R}\Gamma_Z\big(\mathbb{Z}_{D,X}(p+d)\big)$$

qui induit en cohomologie un morphisme de Gysin compatible avec le morphisme de Gysin topologique.

DÉMONSTRATION. Le morphisme s'écrit au niveau des complexes

$$\begin{tikzcd}
 & \mathbb{Z}_Z \arrow[r]\arrow[d] & \Omega^1_Z \arrow[r]\arrow[d] & \cdots \arrow[r] & \Omega^{p-1}_Z \arrow[d] \\
\cdots \arrow[r] & \mathbb{Z}_Z \oplus B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^d_X \arrow[r] & B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^{d+1}_X \arrow[r] & \cdots \arrow[r] & B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^{p-1}_X
\end{tikzcd}$$

La première flèche verticale est donnée par $(\mathrm{id},\mathrm{Bl})$. Les autres sont obtenues par le procédé suivant. Si $Z = Z_1\cap\cdots\cap Z_d$, $B^\infty_{Z|X} = B^\infty_{Z_1|X}\otimes_{\mathcal{O}_X}\cdots\otimes_{\mathcal{O}_X} B^\infty_{Z_d|X}$. On prolonge localement tout élément ω de Ω^i_Z en un élément $\widetilde{\omega}$ de Ω^i_X et ensuite on suit la flèche

$$\Omega^i_X \xrightarrow[\mathrm{Bl}\otimes\mathrm{id}]{} \big(B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^d_X\big)\otimes_{\mathcal{O}_X}\Omega^i_X \xrightarrow[\wedge]{} B^\infty_{Z|X}\otimes_{\mathcal{O}_X}\Omega^{d+i}_X.$$

Deux prolongements de ω diffèrent d'un élément de l'idéal engendré par $f_1, \ldots, f_d, df_1, \ldots, df_d$. Comme $B^{\infty}_{Z_k|X} = j_*\mathcal{O}_{X\setminus Z_k}/\mathcal{O}_X$ et $\{Z\}_{\mathrm{Bl}} = 1/f_1 \otimes \cdots \otimes 1/f_d \; df_1 \wedge \cdots \wedge df_d$, les f_k, comme les df_k, annulent $\{Z\}_{\mathrm{Bl}}$. Le morphisme $\Omega^i_X \longrightarrow B^{\infty}_{Z|X} \otimes_{\mathcal{O}_X} \Omega^{d+i}_X$ est donc bien défini. $\square$

6. Courants normaux, courants rectifiables, courants entiers

Le but de cette section est d'établir l'existence d'une résolution molle explicite du faisceau constant $\mathbb{Z}_X$. Cela permettra ensuite de définir le morphisme de Gysin en cohomologie de Deligne pour une application holomorphe $f : X \longrightarrow Y$ entre variétes compactes. La résolution sera obtenue par des faisceaux de courants rectifiables spéciaux, appelés courants entiers.

Soit X une variété. On note $\mathcal{A}^m_{X,c}$ le faisceau des m-formes différentielles $\mathcal{C}^\infty$ sur X à support compact.

Définition 6.1. *Le faisceau $\mathcal{D}_{X,m}$ des courants de degré m sur X est défini de la manière suivante : pour tout ouvert U de X, $\mathcal{D}_m(U)$ est l'ensemble des applications $T : \mathcal{A}^m_c(U) \longrightarrow \mathbb{C}$ telles que pour tout compact K inclus dans U, il existe $l \in \mathbb{N}$ et $C > 0$ tels que $\forall \alpha \in \mathcal{A}^m_c(U)$ tel que* $\mathrm{supp}(\alpha) \subseteq K$, $| < T, \alpha > | \leq C||\alpha||_{\mathcal{C}^l}$.

Commençons par rappeler les résultats suivants :

Proposition 6.2.

(i) *Si X est une variété réelle $\mathcal{C}^\infty$, le complexe $\left(\mathcal{D}^\bullet_X, d\right)$ est quasi-isomorphe à $\mathbb{C}_X$.*

(ii) *Si X est une variété complexe, pour tout p le complexe $\left(\mathcal{D}^{p,\bullet}_X, \overline{\partial}\right)$ est quasi-isomorphe à Ω^p_X.*

Ces résultats sont classiques et sont les généralisations des lemmes de Poincaré et Dolbeault-Grothendieck pour les courants. On détaille ci-dessous la preuve du point (i) de la proposition 6.2 car on l'adaptera ensuite aux courants rectifiables. Avant cela, précisons les notations.

- On note $\mathcal{D}_{X,m}$ le dual de $\mathcal{A}^m_X$ (m-formes $\mathcal{C}^\infty$), et on pose $\mathcal{D}^k_X = \mathcal{D}_{X,\dim X - k}$.
- Les indices placés en bas correspondent à l'indexation homologique, et les indices placés en haut à l'indexation cohomologique.
- Le degré d'un courant est son degré homologique.

On passe maintenant à la preuve du point (i).

Démonstration. Soit $U = B(0,r)$ la boule de centre 0 et de rayon r de $\mathbb{R}^n$, et soit T un élément de $\mathcal{D}_m(U)$ tel que $dT = 0$. En multipliant T par une fonction plateau, on peut supposer que T est à support compact et que $dT = 0$ dans $B(0,r')$ où $0 < r' < r$. On considère alors le courant S obtenu par la construction du cône :

Soit $h : [0,1] \times B(0,r) \longrightarrow B(0,r)$ définie par $h(t,x) = tx$ et soit $[[0,1]]$ le courant d'intégration sur $[0,1]$. Rappelons que le produit de deux courants est défini par

$$< T_1 \times T_2, \, \omega_1 \otimes \omega_2 >=< T_1, \, \omega_1 >< T_2, \, \omega_2 > \cdot$$

On pose $S = h_*\big([[0,1]] \times T\big)$. Alors sur $B(0,r')$

$$dS = h_*\big(d[[0,1]] \times T + (-1)^n [[0,1]] \times dT\big) = h_*\big(\big[\delta_{\{1\}} - \delta_{\{0\}}\big] \times T\big).$$

On a $\delta_{\{t\}} \times T = i_{t*}X$, où $i_t : \{t\} \times X \longrightarrow [0,1] \times X$ est l'injection naturelle. On en déduit $h_*\big(\delta_{\{t\}} \times T\big) = \big(h \circ i_t\big)_* T$. Pour $t = 1$ on obtient T, pour $t = 0$, on obtient 0, d'où $dS = T$. $\square$

On passe maintenant à la notion de courant normal.

DÉFINITION 6.3.
(i) Soit U un ouvert de $\mathbb{R}^n$ et T un courant de degré m sur U. La masse *de T sur U, notée $M(T)$, est définie par*

$$M(T) = \sup_{\alpha \in \mathcal{A}^m(U),\ ||\alpha|| \leq 1} \langle T, \alpha \rangle.$$

(ii) Soit X une variété différentiable paracompacte. Un courant T est dit de masse localement finie *sur X si pour tout x dans X il existe un voisinage U_x de x difféomorphe à un ouvert de $\mathbb{R}^n$ tel que $M(T_{|U_x})$ soit finie.*

Les courants de masse localement finie forment un faisceau.

Le théorème de représentation de Riesz permet d'identifier localement les courants de degré m et de masse localement finie avec des mesures de Radon vectorielles sur X. Plus précisément, si T est un tel courant, il s'écrit dans un ouvert de coordonnées $T = \sum_{|I|=n-m} \mu_I \, dx^{|I|}$, où les μ_I sont des mesures de Radon sur U.

DÉFINITION 6.4.
(i) Un courant T sur un ouvert U de $\mathbb{R}^n$ est normal *si T et dT sont de masse finie.*

(ii) Soit X une variété differentiable paracompacte et T un courant sur X. On dit que T est localement normal *si pour tout point x de X il existe un ouvert de coordonnées U_x contenant x tel que $T_{|U_x}$ soit normal.*

Les courants localement normaux forment un faisceau, noté $\mathcal{N}_m^{\text{loc}}$.

DÉFINITION 6.5.
(i) Un sous ensemble M de $\mathbb{R}^n$ est m-rectifiable *si M est inclus dans un ensemble de la forme $N \cup \Big(\bigcup_{j=1}^{\infty} f_j(A_j) \Big)$ où*

- *La mesure de Hausdorff m-dimensionnelle de N est nulle.*
- *Pour tout j, A_j est un sous ensemble de $\mathbb{R}^m$ et les applications $f_j : A_j \longrightarrow M$ sont lipschitziennes.*

(ii) Soit X une variété paracompacte. Un sous-ensemble M de X est dit localement m-rectifiable *si pour tout point x de X il existe un ouvert de coordonnées U_x contenant x tel que $M \cap U_x$ soit m-rectifiable.*

Soit (X, g) une variété riemannienne et $\mathcal{H}^m_{X,g}$ la mesure de Hausdorff m-dimensionnelle associée sur X. Si M est un sous-ensemble de X localement rectifiable, on note $\mu^m_{M,g}$ la mesure trace de $\mathcal{H}^m_{X,g}$ sur M. C'est une mesure de Radon.

EXEMPLE 6.6.

(i) *Toute sous-variété de dimension m de $\mathbb{R}^n$ est localement rectifiable.*

(ii) *Soit (x_n) un suite de points de $\mathbb{R}^2$ et (r_n) une suite de nombres positifs telle que la série de terme général r_n converge. Si $D(x_n, r_n)$ désigne le disque de $\mathbb{R}^2$ centré en x_n et de rayon r_n, l'ensemble $\bigcup_{n \geq 0} \partial D(x_n, r_n)$ est rectifiable.*

Les ensembles rectifiables sont caractérisés par le fait qu'ils admettent presque partout (au sens de la mesure trace de $\mathcal{H}^m$) un espace tangent de dimension m. Introduisons la définition suivante :

DEFINITION 6.1. Soit $x \in \mathbb{R}^n$, μ une mesure sur une boule $B(x,r)$ et ν une mesure non nulle sur $\mathbb{R}^n$. On dit que μ est *tangente* à ν en x s'il existe des suites numériques $(\rho_j)_{j\geq 1}$ et $(c_j)_{j\geq 1}$ vérifiant

- $\lim_{j\to\infty} \rho_j = +\infty$,
- si $h_j : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ est l'homothétie de centre x et de rapport ρ_j, pour toute fonction $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ continue à support compact,

$$\lim_{j\to\infty} c_j \int_{B(x,r)} f \circ h_j \, d\mu = \int_{\mathbb{R}^n} f \, d\nu.$$

REMARQUE 6.7.

(i) *Si j est assez grand, supp$(f \circ h_j) \subseteq B(x,r)$ et le membre de gauche est bien défini.*

(ii) *La définition n'est pas symétrique en μ et ν. En effet, si $0 < r' < r$ et si $\mathbb{1}_{B(x,r')}$ désigne la fonction charactéristique de la boule $B(x,r')$, μ est tangente à ν en x si et seulement si $\mathbb{1}_{B(x,r')}\,\mu$ est tangente à ν en x. Cependant, si μ est tangente en x à deux mesures ν_1 et ν_2, il existe $c > 0$ tel que $\nu_2 = c\nu_1$.*

EXEMPLE 6.8.

(i) *Soit $C = \{(x,y) \in \mathbb{R}^2 \,/\, 0 \leq x \leq 1$ et $y = x^2\}$ et μ la mesure d'intégration sur C. Si f est à support compact dans $\mathbb{R}^2$ et $\rho \gg 0$,*

$$\int_{[0,1]} f \circ h_\rho \, d\mu = \int_0^1 f(\rho x, \rho x^2)\, dx = \frac{1}{\rho}\int_0^\rho f\left(u, \frac{u^2}{\rho}\right) du$$

donc $\displaystyle\lim_{j\to\infty} \rho \int (f \circ h_\rho)\, d\mu = \int_0^\infty f(u,0)\, du$. La mesure μ est donc tangente en $(0,0)$ à la mesure d'intégration sur $\mathbb{R}^+ \times \{0\}$.

(ii) *Si M est une sous-variété de $\mathbb{R}^n$ et $x \in M$, la mesure d'intégration sur M est tangente à la mesure d'intégration sur l'espace tangent T_xM, vu comme sous-espace affine de $\mathbb{R}^n$.*

REMARQUE 6.9. *Si $\phi : B(x,r) \longrightarrow B(x,r')$ est un difféomorphisme local en x et si μ est tangente à ν en x, la mesure $\phi_*\mu$ est tangente à $D\phi_{x*}\nu$. Si ν est une mesure linéaire, c'est-à-dire si $\nu(f) = \int_{W_x} f(x)\, d\lambda(x)$ où W_x est un sous-espace affine de $\mathbb{R}^n$ passant par x et $d\lambda$ une mesure de Lebesgue non nulle sur W_x, alors $D\phi_{x*}\nu$ est également linéaire. Le fait qu'une mesure μ soit tangente à une mesure linéaire en x ne dépend donc pas des coordonnées locales utilisées au voisinage de x.*

THÉORÈME 6.10. *Soit M un sous-ensemble m-rectifiable de $\mathbb{R}^n$. Pour μ_M^m-presque tout x dans M, μ_M^m admet une mesure linéaire tangente en x donnée par un sous-espace affine T_xM de dimension m de $\mathbb{R}^n$.*

Pour la démonstration, voir [**Fe**], 3.2.25.

On souhaite maintenant définir des courants associés aux ensembles rectifiables par intégration.

Commençons par le cas le plus simple. Soit M une sous-variété de dimension m de $\mathbb{R}^n$. Pour pouvoir intégrer sur M une m-forme, il faut disposer d'une orientation sur M, qui peut s'interpréter comme une application continue $\xi : M \longrightarrow \bigwedge^m TM$ telle que $||\xi|| = 1$. On définit alors le courant d'intégration $[M]$ sur X par

$$\langle [M], \alpha \rangle = \int_M \langle \alpha, \xi \rangle \, d\mu_M^m.$$

Ceci motive la définition suivante :

DÉFINITION 6.11. *Soit X une variété paracompacte munie d'une metrique riemannienne g.*

(i) *Un courant T* localement rectifiable *sur X est un courant de la forme*

$$\langle T, \alpha\rangle = \int_M \langle \alpha, \xi\rangle\, \Theta\, d\mu^m_{M,g}$$

où

- *$M \subseteq X$ est localement m-rectifiable,*
- *l'application $\xi\colon M \longrightarrow \bigwedge^m TX$ est une section $\mu^m_{M,g}$-mesurable de $\bigwedge^m TM$ tel que $||\xi||_g = 1$ $\mu^m_{M,g}$-presque partout,*
- *l'application $\Theta\colon M \longrightarrow \mathbb{Z}$ est $\mu^m_{M,g}$-mesurable.*

(ii) *Un courant T est* localement entier *si T et dT sont localement rectifiables.*

La deuxième condition du point (i) ci-dessus signifie que pour tout point x de M, ξ est en coordonnées locales une application de $M \cap U_x$ dans $\bigwedge^m \mathbb{R}^n$ telle que

- pour $\mu^m_{M,g}$-presque tout y dans $M \cap U_x$, $\xi(y) \in \bigwedge^m \overline{T_y M}$, où $\overline{T_y M}$ est le sous-espace vectoriel associé au sous-espace affine $T_y M$.
- pour $\mu^m_{M,g}$-presque tout y dans $M \cap U_x$, $||\xi(y)|| = 1$, où $\mathbb{R}^n \simeq T_y X$ est muni du produit scalaire g_y.

Les courants localement rectifiables et localement entiers seront notés $\mathcal{R}^{\mathrm{loc}}_m$ et $\mathcal{I}^{\mathrm{loc}}_m$.

REMARQUE 6.12.

(i) *On peut en fait choisir n'importe quel sous-anneau A de $\mathbb{C}$ et prendre Θ à valeurs dans A. On parle alors de courant rectifiables à coefficients dans A. Nous n'utiliserons que les cas $A = \mathbb{Z}$ ou $A = \mathbb{Q}$.*

(ii) *Si T est localement rectifiable, $-T$ aussi. Si T et T' sont localement rectifiables, on a avec des notations évidentes*

$$\langle T + T', \alpha\rangle = \int_M \langle \alpha, \xi\rangle\, \Theta\, d\mu^m_{M,g} + \int_{M'} \langle \alpha, \xi'\rangle\, \Theta'\, d\mu^m_{M,g} = \int_{M\cup M'} \langle \alpha, \xi\, \Theta\, \mathbb{1}_M + \xi'\, \Theta'\, \mathbb{1}_{M'}\rangle\, d\mu^m_{M,g}$$

Sur $M \cap M'$, on peut écrire $\xi' = \varepsilon\xi$, où $\varepsilon\colon M \cap M' \longrightarrow \{-1, 1\}$ est $\mu^m_{M,g}$-mesurable. Alors $\xi\Theta + \xi'\Theta' = \xi(\Theta + \varepsilon\,\Theta')$. Si on pose

$$\widetilde{\xi} = \xi\, \mathbb{1}_{M'^c} + \xi'\, \mathbb{1}_{M'} \quad et \quad \widetilde{\Theta} = \Theta\, \mathbb{1}_{M\cap(M\cap M')^c} + (\Theta + \varepsilon\Theta')\, \mathbb{1}_{M\cap M'} + \Theta'\, \mathbb{1}_{M'\cap(M\cap M')^c},$$

on a $\xi\Theta\, \mathbb{1}_M + \xi'\Theta'\, \mathbb{1}_{M'} = \widetilde{\xi}\,\widetilde{\Theta}$, ce qui montre que $T + T'$ est rectifiable.

(iii) *Si U est un ouvert de X et si T est rectifiable sur X, alors $T_{|U}$ est rectifiable sur U.*

PROPOSITION 6.13. *Soit X une variété paracompacte. Le préfaisceau $U \longrightarrow \mathcal{R}^{loc}_m(U)$ est un faisceau mou sur X.*

DÉMONSTRATION. Comme $\mathcal{D}_m(U)$ est un faisceau, le préfaisceau $U \longrightarrow \mathcal{R}_m(U)$ satisfait l'axiome d'unicité. Vérifions l'axiome de recollement. Soit U_i une famille d'ouverts de X. On suppose donné pour tout i un courant rectifiable T_i sur U_i tel que pout tout couple (i, j) T_i et T_j coincident sur U_{ij}. Écrivons $T_i = \int_{M_i} \langle\, .\, , \xi_i\rangle\, \Theta_i\, d\mu^m_{M,g}$. Comme $T_{i|U_{ij}} = T_{j|U_{ij}}$, on a

$$\begin{cases} M_i \cap U_{ij} = M_j \cap U_{ij} & \text{On note cet ensemble } M_{ij} \\ \xi_i \Theta_{i_{|M_{ij}}} = \xi_j \Theta_{j_{|M_{ij}}} & \mu^m_{M,g} \text{ presque partout.} \end{cases}$$

Il existe donc M localement rectifiable tel que pour tout i, $M_{|U_i} = M_i$. Définissons Θ par $\Theta_{|U_i} = |\Theta_i|$. Comme $||\xi_i|| = 1$, $|\Theta_i|_{|U_{ij}} = |\Theta_j|_{|U_{ij}}$, et la définition a un sens. On peut écrire $\Theta_i = \varepsilon_i |\Theta_i|$, où ε_i est $\mu^m_{M,g}$-mesurable. On a alors $\xi_i \, \varepsilon_{i|U_{ij}} = \xi_j \, \varepsilon_{j|U_{ij}}$. On peut donc recoller ces fonctions en une fonction globale ξ, et

$$\xi_{U_i} \, \Theta_{U_i} = \Big(\xi_i \, \varepsilon_i \, |\Theta_i|\Big)_{|U_{ij}} = \big(\xi_i \, \Theta_i\big)_{|U_{ij}}.$$

Le courant localement rectifiable T défini par le triplet (M, ξ, Θ) vérifie donc $T_{|U_i} = T_i$.

Considérons maintenant un fermé Z de X et un courant T localement rectifiable sur un voisinage U de Z. Écrivons $T = (M, \, \xi, \, \Theta)$. M est localement rectifiable dans U mais pas nécessairement dans X (c.f. Remarque 6.14). Soit U' un voisinage de Z tel que $\overline{U'} \subseteq U$. Soit $M' = M \cap U'$, $\xi' = \xi_{|M'}$, $\Theta' = \Theta_{|U'}$. Alors $T_{|U'}$ est associé au triplet $(M', \, \xi', \, \Theta')$; M' est localement rectifiable dans U' mais également dans X. Le triplet $(M', \, \xi', \, \Theta')$ définit donc un courant S localement rectifiable sur X qui vérifie $S_{|U'} = T_{|U'}$. □

Remarque 6.14. *Le faisceau $\mathcal{R}^{loc}_m$ n'est pas flasque. En effet, si U est un ouvert de X, un sous ensemble A de U, localement rectifiable dans U, peut ne pas être localement rectifiable dans X. Par exemple, l'image de la courbe*

$$[0, 1[\longrightarrow \mathbb{C}, \qquad t \longmapsto t \, \exp\big(i/(1-t)\big)$$

est localement rectifiable dans la boule unité ouverte de $\mathbb{R}^2$, mais pas dans $\mathbb{R}^2$ car sa mesure de Hausdorff 1-dimensionnelle est infinie au voisinage de tout point de la sphère unité de $\mathbb{R}^2$.

Proposition 6.15. *Soit X une variété compacte. Le faisceau $\mathcal{I}^{\rm loc}_m$ est mou sur X.*

Démonstration. La preuve est plus délicate que pour les courants rectifiables. En effet, en reprenant les notations de la proposition 6.13, si T est localement entier sur un voisinage U de Z et si U' est un voisinage de Z relativement compact dans U, $T_{|U'}$ est localement rectifiable dans X mais on ne peut rien dire a priori de $d(T_{|U'})$.

Pour remédier à ce problème, on utilise la théorie du slicing pour les courants normaux. Soit $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ lipschitzienne, $r_0 \in \mathbb{R}$ et T un courant normal sur un voisinage de l'ensemble des points x de $\mathbb{R}^n$ tels que $f(x) \leq r_0$. Si $r < r_0$, la tranche $< T, f, r >$ est définie par

$$< T, f, r >= dT_{|\{x/f(x) \leq r\}} - d\big[T_{|\{x/f(x) \leq r\}}\big].$$

Soit $d\lambda$ la mesure de Lebesgue sur $\mathbb{R}$. Le point essentiel est que pour $d\lambda$-presque tout $r < r_0$, $< T, f, r >$ est de masse finie ([**Fe**] 4.2.1).

Dans notre cas, X est une variété différentiable compacte que l'on peut supposer plongée dans $\mathbb{R}^n$ par le theorème de Whitney. Soit W un voisinage tubulaire de X dans $\mathbb{R}^n$, $\pi : W \longrightarrow X$ la projection associée et $i : X \longrightarrow W$ l'injection. Le courant $i_* T$ est alors localement entier en tant que courant sur $\pi^{-1}(U)$.

On applique ce qui précède à la fonction $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ définie par $f(x) = \mathrm{dist}(x, F)$, qui est 1-lipschitzienne. Si $r_0 > 0$ est suffisamment petit, $i_* T$ est normal sur un voisinage de l'ensemble $\{x \in \mathbb{R}^n / f(x) < r_0\}$. Pour $r \in]0, r_0[$ en dehors d'un ensemble de mesure de Lebesgue nulle, le tranche $< i_* T, f, r >$ est donc un courant de masse bornée.

Ceci implique que $i_* T_{|\{x \in \mathbb{R}^n / f(x) \leq r\}}$ est normal. Par suite $T_{|\{x \in \mathbb{R}^n / f(x) \leq r\}}$ est normal, car π est propre sur le support de $T_{|\{x \in \mathbb{R}^n / f(x) \leq r\}}$ et $\pi \circ i = \mathrm{id}$. Le courant $T_{|\{x \in \mathbb{R}^n / f(x) \leq r\}}$ est un courant rectifiable et normal sur U, donc entier ([**Fe**] 4.2.16 (2)). Soit $F = \{x \in \mathbb{R}^n / f(x) \leq r\} \cap X \subseteq U$.

Alors F est un fermé qui contient un voisinage de Z et $T_{|F}$ est un courant localement entier sur U. Le courant $T_{|F}$ est alors localement entier sur X, ce qui termine la démonstration. □

PROPOSITION 6.16. *Soit $f : X \longrightarrow Y$ une application lipschitzienne propre. Alors le morphisme $f_* : \mathcal{D}_{m,X} \longrightarrow \mathcal{D}_{m,Y}$ envoie $\mathcal{R}^{\mathrm{loc}}_{m,X}$ dans $\mathcal{R}^{\mathrm{loc}}_{m,Y}$.*

Pour la preuve, voir [**Fe**] 4.1.25 et 4.1.26.

REMARQUE 6.17. *Comme $d(f_*T) = f_*\,dT$, les courants localement entiers sont préservés par f_*. Le morphisme $f_* : \mathcal{D}_{m,X} \longrightarrow \mathcal{D}_{m,Y}$ envoie $\mathcal{I}^{\mathrm{loc}}_{m,X}$ dans $\mathcal{I}^{\mathrm{loc}}_{m,Y}$.*

Pour terminer cette section, on montre le résultat suivant :

PROPOSITION 6.18. *Si n est la dimension de X, on a la suite exacte*

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow \mathcal{I}^{\mathrm{loc}}_{n,X} \xrightarrow{d} \mathcal{I}^{\mathrm{loc}}_{n-1,X} \longrightarrow \cdots \longrightarrow \mathcal{I}^{\mathrm{loc}}_{0,X} \longrightarrow 0.$$

DÉMONSTRATION. Il faut tout d'abord montrer un lemme de Poincaré local pour les faisceaux $\mathcal{I}^{\mathrm{loc}}_k$. On reprend la preuve faite au début de la section 6. Seuls quelques points sont à justifier.

– En appliquant l'argument de slicing utilisé dans la démonstration de la proposition 6.15 à la fonction $u(x) = ||x||$, on obtient que pour presque tout r' dans $]0, r[$, $T_{|\overline{B}(0,r')}$ est entier sur $B(0, r)$.

– Il faut montrer que le courant S défini par $S = h_*\big([[0,1]] \times T\big)$ est un courant localement entier. Pour cela, on voit facilement que $[[0,1]] \times T$ est localement rectifiable dans $\mathbb{R} \times B(0, r)$ associé au triplet

$$[0,1] \times M, \qquad (t,x) \longrightarrow \frac{\partial}{\partial t} \wedge \xi(x)\ , \qquad \widetilde{\Theta}(t,x) = \Theta(x).$$

Comme $\mathrm{supp}\big([[0,1]] \times T\big)$ est compact, $h_*\big([[0,1]] \times T\big)$ est localement rectifiable par la proposition 6.16. S est donc localement rectifiable. Comme $dS = T$, S est localement entier.

Il reste maintenant à montrer le noyau de $d : \mathcal{I}^{\mathrm{loc}}_{n,X} \longrightarrow \mathcal{I}^{\mathrm{loc}}_{n-1,X}$ s'identifie canoniquement à $\mathbb{Z}_X$. Si T appartient à $\mathcal{I}^{\mathrm{loc}}_{n,X}$, une orientation de X sur U fixée permet d'écrire

$$\langle T, \alpha\, dx_1 \wedge \cdots \wedge dx_n \rangle = \int_M \Theta\, \alpha\, dx^1 \wedge \cdots \wedge dx^n,$$

où $\Theta : M \longrightarrow \mathbb{Z}$ est mesurable. On peut d'autre part identifier T a une distribution sur $U = B(0, r)$. Dans ce cas $\partial T/\partial x_i = 0$ pour tout i et T est donc constante. Il existe donc $c \in \mathbb{C}$ tel que

$$\langle T, \alpha\, dx_1 \wedge \cdots \wedge dx_n \rangle = c \int_M \alpha\, dx^1 \wedge \cdots \wedge dx^n.$$

On en déduit $\Theta = c$ $\mu_{M,g}$-presque partout. □

Les courants localement entiers permettent de travailler avec une résolution molle de $\mathbb{Z}_X$ stable par image directe. Ce formalisme est donc particulièrement bien adapté aux morphismes de Gysin.

Soit $f : X \longrightarrow Y$ une application lipschitzienne propre. Comme les faisceaux $\mathcal{I}_m^{\mathrm{loc}}$ sont mous, pour $i \geq 1$, $R^i f_*(\mathcal{I}_m^{\mathrm{loc}}) = 0$ et par suite $Rf_*\big(\mathcal{I}_{d_X,X}^{\mathrm{loc}} \longrightarrow \mathcal{I}_{d_X-1,X}^{\mathrm{loc}} \longrightarrow \cdots \longrightarrow \mathcal{I}_{0,X}^{\mathrm{loc}} \big)$ est quasi-isomorphe à

$$f_*\mathcal{I}_{d_X,X}^{\mathrm{loc}} \longrightarrow f_*\mathcal{I}_{d_X-1,X}^{\mathrm{loc}} \longrightarrow \cdots \longrightarrow f_*\mathcal{I}_{0,X}^{\mathrm{loc}} .$$

Ce complexe s'envoie par push-forward sur le complexe $\mathcal{I}_{d_X,Y}^{\mathrm{loc}} \longrightarrow \mathcal{I}_{d_X-1,Y}^{\mathrm{loc}} \longrightarrow \cdots \longrightarrow \mathcal{I}_{0,Y}^{\mathrm{loc}}$ et par suite sur le complexe $\mathcal{I}_{d_Y,Y}^{\mathrm{loc}} \longrightarrow \mathcal{I}_{d_Y-1,Y}^{\mathrm{loc}} \longrightarrow \cdots \longrightarrow \mathcal{I}_{0,Y}^{\mathrm{loc}}[\dim Y - \dim X]$. Ceci fournit dans la catégorie dérivée un morphisme $Rf_*\mathbb{Z}_X \longrightarrow \mathbb{Z}_Y[\dim Y - \dim X]$ qui induit en cohomologie le morphisme de Gysin. En effet, si $\mathcal{C}_m^{\mathrm{sing}}(X)$ et $\mathcal{C}_m^{\mathrm{sing}}(Y)$ sont les m-cochaînes singulières lipschitziennes sur X et Y, on a un diagramme commutatif :

$$\begin{tikzcd} \mathcal{C}_m^{\mathrm{sing}}(X) \arrow[r] \arrow[d, "f_*"'] & \mathcal{I}_m^{\mathrm{loc}}(X) \arrow[d, "f_*"] \\ \mathcal{C}_m^{\mathrm{sing}}(Y) \arrow[r] & \mathcal{I}_m^{\mathrm{loc}}(Y) \end{tikzcd}$$

7. Morphisme de Gysin en cohomologie de Deligne

On commence par fixer les notations. Si X est une variété analytique complexe lisse, on introduit les faisceaux suivants :

- $\mathcal{D}_X^i$ est le faisceau des courants de degré $2\dim X - i$.
- $\mathcal{D}_{X,\mathbb{Z}}^i = \mathcal{I}_{2\dim X-i,X}^{\mathrm{loc}}$ (voir section 6) est le faisceau des courants entiers de degré $2\dim X - i$.
- $F^p\mathcal{D}_X^i = \bigoplus\limits_{\substack{r+s=i \\ r\geq p}} \mathcal{D}_X^{r,s}$.

On a alors les résultats suivants :

1. $\mathcal{D}_{X,\mathbb{Z}}^\bullet$ est une résolution acyclique de $\mathbb{Z}_X$.

2. $F^p\mathcal{D}_X^\bullet$ est un complexe acyclique quasi-isomorphe à $F^p\Omega_X^\bullet$.

3. $\mathcal{D}_X^\bullet$ est une résolution acyclique de $\mathbb{C}_X$.

De plus, si $f : X \longrightarrow Y$ est holomorphe propre et si $d = \dim Y - \dim X$, on a trois morphismes d'images directe donnés par le push-forward des courants :

$$f_*\mathcal{D}_{X,\mathbb{Z}}^\bullet \xrightarrow{f_*} \mathcal{D}_{Y,\mathbb{Z}}^\bullet[2d], \qquad f_*F^p\mathcal{D}_X^\bullet \xrightarrow{(2i\pi)^d f_*} F^{p+d}\mathcal{D}_X^\bullet[2d], \qquad f_*\mathcal{D}_X^\bullet \xrightarrow{(2i\pi)^d f_*} \mathcal{D}_Y^\bullet[2d].$$

Rappelons que $\mathbb{Z}_{D,X}(p) \simeq \mathrm{Mc}\big(\mathbb{Z}_X \oplus \Omega_X^{\bullet \geqslant p} \longrightarrow \Omega_X^\bullet\big)[-1]$. Par suite, $\mathbb{Z}_{D,X}(p)$ est quasi-isomorphe à $\mathrm{Mc}\big(\mathcal{D}_{X,\mathbb{Z}}^\bullet \oplus F^p\mathcal{D}_X^\bullet \longrightarrow \mathcal{D}_X^\bullet\big)[-1]$. Ainsi $Rf_*\mathbb{Z}_{D,X}(p) \simeq \mathrm{Mc}\big(f_*\mathcal{D}_{X,\mathbb{Z}}^\bullet \oplus f_*F^p\mathcal{D}_X^\bullet \longrightarrow \mathcal{D}_X^\bullet\big)[-1]$. On en déduit un morphisme dans la catégorie dérivée

$$Rf_*\mathbb{Z}_{D,X}(p) \longrightarrow \mathbb{Z}_{D,Y}(p+d)[2d].$$

DÉFINITION 7.1. *Le morphisme induit en cohomologie* $H_{Del}^*(X, \mathbb{Z}(p)) \longrightarrow H_{Del}^{*+2d}(Y, \mathbb{Z}(p+d))$ *est noté* f_**. C'est le* morphisme de Gysin *associé à* f*.*

Dans la section 5, on a déjà construit un morphisme de Gysin dans le cas d'une immersion $i_Z : Z \hookrightarrow X$ à l'aide des faisceaux $B_{Z|X}$.

Proposition 7.2. *Supposons que Z est une sous-variété lisse de X de codimension d. Alors le morphisme*

$$H^*_{\mathrm{Del}}(X, \mathbb{Z}(p)) \longrightarrow H^{*+2d}_{\mathrm{Del},Z}(X, \mathbb{Z}(p+d)) \longrightarrow H^{*+2d}_{\mathrm{Del}}(X, \mathbb{Z}(p+d))$$

construit dans la section 5 coïncide avec le morphisme de Gysin i_{Z*}.

Démonstration. Soit $\mathcal{B}_X$ le faisceau des hyperfonctions sur X [**Scha**]. C'est un faisceau flasque qui contient $\mathcal{D}_X$. Si $\mathcal{B}^{p,q}_X = \mathcal{B}_X \otimes_{\mathcal{O}_X} \Omega^{p,q}_X$, $\mathcal{B}^{p,\bullet}_X$ est une résolution flasque de Ω^p_X [**Ko**]. On a donc

$$B^\infty_{Z|X}[-d] = \mathcal{R}\Gamma_Z\big(\mathcal{O}_X\big) \simeq \mathcal{R}\Gamma_Z\big(\mathcal{B}^{0,\bullet}_X\big) \simeq \Gamma_Z\big(\mathcal{B}^{0,\bullet}_X\big) \simeq \mathcal{H}^d\Big(\Gamma_Z\big(\mathcal{B}^{0,\bullet}_X\big)\Big)[-d].$$

Donc pour tout p, $B^\infty_{Z|X} \otimes_{\mathcal{O}_X} \Omega^p_X \simeq \mathcal{H}^d\Big(\Gamma_Z\big(\mathcal{B}^{p,\bullet}_X\big)\Big)$.

Lemme 7.3. *L'isomorphisme $\mathcal{H}^d\Big(\Gamma_Z\big(\mathcal{B}^{d,\bullet}_X\big)\Big) \longmapsto B^\infty_{Z|X} \otimes_{\mathcal{O}_X} \Omega^d_X$ envoie $(2i\pi)^d\,[Z]$ sur $\big\{Z\big\}_{|\mathrm{Bl}}$, où $[Z]$ est le courant d'intégration sur Z.*

Démonstration. On considère le complexe de Koszul $K^\bullet$ associé aux paramètres locaux $z_1, \ldots, z_d$ définissant Z. Alors

$$\mathcal{RH}om_{\mathcal{O}_X}\Big(\mathcal{O}_X/_{\mathcal{J}_Z}, \Omega^d_X\Big) \simeq \mathcal{H}om_{\mathcal{O}_X}\big(K^\bullet, \Omega^d_X\big) \simeq \mathcal{H}om_{\mathcal{O}_X}\big(K^\bullet, \mathcal{D}^{d,\bullet}_X\big).$$

En degré k, ce complexe est égal à $\bigoplus_{i=0}^{\min(d,k)} \mathcal{D}^{d,k-i}_X \otimes_{\mathbb{C}} \bigwedge^i \mathbb{C}^d$, la différentielle étant donnée par

$$\delta(T\, e_1 \wedge \ldots \wedge e_l) = \sum_{i=0}^{l} (-1)^{i-1} z_i\, T\, e_1 \wedge \ldots \wedge e_i \wedge \ldots \wedge e_l + (-1)^{\mathrm{d}^\circ(T)}\, \overline{\partial}\, T\, e_1 \wedge \ldots \wedge e_l$$

où $\mathrm{d}^\circ(T)$ désigne le deuxième indice du bidegré de T. Montrons que les éléments $\big(dz_1 \wedge \ldots \wedge dz_d\big)\, e_1 \wedge \ldots \wedge e_d$ et $(2i\pi)^d[Z]$ diffèrent d'un bord. On calcule

$$\big\{dz_1 \wedge \ldots \wedge dz_d\big\}\, e_1 \wedge \ldots \wedge e_d = \delta\Big(\Big\{\frac{dz_1}{z_1} \wedge dz_2 \ldots \wedge dz_d\Big\}\, e_2 \wedge \ldots \wedge e_d\Big) + \overline{\partial}\Big(\frac{dz_1}{z_1} \wedge \ldots \wedge dz_d\Big)\, e_2 \wedge \ldots \wedge e_d.$$

On a $\overline{\partial}\Big(\dfrac{dz_1}{z_1} \wedge \ldots \wedge dz_d\Big) = 2i\pi[Z_1] \wedge dz_2 \wedge \ldots \wedge dz_d$, où $Z_1 = \{z_1 = 0\}$. On recommence :

$$\begin{aligned} 2i\pi\, \big([Z_1] \wedge dz_1 \ldots \wedge dz_d\big)\, e_2 \wedge \ldots \wedge e_d &= (2i\pi)\, i_{Z_1 *}\big(dz_2 \wedge \ldots \wedge dz_d\big)\, e_2 \wedge \ldots \wedge e_d \\ &= 2i\pi\Big(-\delta\Big(i_{Z_1 *}\Big(\frac{dz_2}{z_2} \wedge \ldots \wedge dz_d\Big)\, e_3 \wedge \ldots \wedge e_d\Big) + \overline{\partial}\Big(i_{Z_1 *}\Big(\frac{dz_2}{z_2} \wedge \ldots \wedge dz_d\Big)\Big)\, e_3 \wedge \ldots \wedge e_d\Big) \end{aligned}$$

et $\overline{\partial}\Big(i_{Z_1 *}\big(\dfrac{dz_2}{z_2} \wedge \ldots \wedge dz_d\big)\Big) = (2i\pi)\, i_{Z_2 *}\,(dz_3 \wedge \ldots \wedge dz_d)$, où $Z_2 = \{z_1 = 0\} \cap \{z_2 = 0\}$. On arrive en définitive à $(2i\pi)^d[Z_d]$, c'est-à-dire $(2i\pi)^d[Z]$. On obtient ainsi la compatibilité. □

On peut réaliser le quasi-isomorphisme $B^\infty_{Z|X}[-d] \otimes_{\mathcal{O}_X} \Omega^{\bullet \geqslant p}_X \xrightarrow{\simeq} \Gamma_Z\big(F^p \mathcal{B}^\bullet_X\big)$ au niveau des complexes de la manière suivante :

$$B^\infty_{Z|X}[-d] \otimes_{\mathcal{O}_X} \Omega^{\bullet \geqslant p}_X \xrightarrow{\simeq} \mathcal{H}^d\Big(\Gamma_Z\big(\mathcal{B}^{0,\bullet}_X\big)\Big) \otimes_{\mathcal{O}_X} \Omega^{\bullet \geqslant p}_X \xrightarrow{\simeq} \mathcal{H}^d\Big(\Gamma_Z\big(F^p\, \mathcal{B}^\bullet_X\big)\Big)$$

$$\xleftarrow{\simeq} \tau^{\leqslant d}\,\Gamma_Z\big(F^p\,\mathcal{B}_X^\bullet\big) \xrightarrow{\simeq} \Gamma_Z\big(F^p\,\mathcal{B}_X^\bullet\big)$$

Montrons qu'on a un diagramme commutatif :

$$\begin{tikzcd}
\Omega_Z^{\bullet\geqslant p} \arrow[r] \arrow[d] & \Omega_Z^{\bullet\geqslant p} \arrow[r] \arrow[d] & \Omega_Z^{\bullet\geqslant p} \arrow[r] \arrow[d] & F^p\,\mathcal{B}_Z^\bullet \arrow[d, "(2i\pi)^d\, i_{Z*}"'] \\
B^\infty_{Z|X}[-d]\otimes_{\mathcal{O}_X}\Omega_X^{p+d} \arrow[r, "\simeq"'] & \mathcal{H}^d\big(\Gamma_Z\big(F^{p+d}\mathcal{B}_X^\bullet\big)\big)[2d] & \tau^{\leqslant d}\,\Gamma_Z\big(F^{p+d}\,\mathcal{B}_X^\bullet\big)[2d] \arrow[l, "\simeq"] \arrow[r, "\simeq"'] & \Gamma_Z\big(F^{p+d}\,\mathcal{B}_X^\bullet\big)[2d]
\end{tikzcd}$$

Ceci revient par le lemme 7.3 à montrer que si $\omega\in\Omega_Z^i$ et si $\widetilde{\omega}$ est une extension locale de ω, $(2i\pi)^d\,[Z]\wedge\widetilde{\omega}=(2i\pi)^d i_{Z*}\omega$, ce qui est évident. En prenant les cônes, on obtient la compatibilité entre les deux constructions. $\square$

On peut maintenant établir le résultat suivant :

PROPOSITION 7.1.

(i) *Le morphisme f_* est fonctoriel et satisfait la formule de projection.*

(ii) *Soit $q\colon E\longrightarrow X$ une submersion holomorphe et Y une sous-variété de X (E, X et Y ne sont pas nécessairement compacts). Soit F le produit fibré de Y et E au dessus de X. Alors, en utilisant les notations données par le diagramme suivant*

$$\begin{tikzcd}
F \arrow[r, hook, "i_F"] \arrow[d, "p"'] & E \arrow[d, "q"] \\
Y \arrow[r, hook, "i_Y"'] & X
\end{tikzcd}$$

on a $q^*i_{Y*}=i_{F*}p^*$.

DÉMONSTRATION. (i) La fonctorialité est évidente. Montrons la formule de projection. On utilise le morphisme Δ construit à la section 4. Montrons que le diagramme de complexes suivant est commutatif :

$$\begin{tikzcd}
Rf_*\Big[\mathrm{Mc}\Big(\mathcal{D}^\bullet_{X,\mathbb{Z}}\oplus F^p\mathcal{D}^\bullet_X\longrightarrow\mathcal{D}^\bullet_X\Big)[-1]\otimes_{\mathbb{Z}}f^{-1}\mathbb{Z}_{D,Y}(q)\Big] \arrow[r, "f_*\otimes\mathrm{id}"] \arrow[d, "\mathrm{id}\otimes f^*"'] & \mathrm{Mc}\Big(\mathcal{D}^\bullet_{Y,\mathbb{Z}}\oplus F^{p+d}\mathcal{D}^\bullet_Y\longrightarrow\mathcal{D}^\bullet_Y\Big)[-1]\,[2d]\otimes_{\mathbb{Z}}\mathbb{Z}_{D,Y}(q) \arrow[dd, "\Delta_Y"] \\
Rf_*\Big[\mathrm{Mc}\Big(\mathcal{D}^\bullet_{X,\mathbb{Z}}\oplus F^p\mathcal{D}^\bullet_X\longrightarrow\mathcal{D}^\bullet_X\Big)[-1]\otimes_{\mathbb{Z}}\mathbb{Z}_{D,X}(q)\Big] \arrow[d, "\Delta_X"'] & \\
Rf_*\Big[\,\mathrm{Mc}\Big(\mathcal{D}^\bullet_{X,\mathbb{Z}}\oplus F^{p+q}\mathcal{D}^\bullet_X\longrightarrow\mathcal{D}^\bullet_X\Big)[-1]\,\Big] \arrow[r, "f_*"'] & \mathrm{Mc}\Big(\mathcal{D}^\bullet_{Y,\mathbb{Z}}\oplus F^{p+q+d}\mathcal{D}^\bullet_Y\longrightarrow\mathcal{D}^\bullet_Y\Big)[-1]\,[2d]
\end{tikzcd}$$

On part d'un élément $(S,T\oplus U)\otimes y$ et on distingue les différents cas :

1. $\mathrm{d}^\circ(y) = 0$.

– En passant par le coin en bas à gauche du diagramme, on obtient :

$$(S, T \oplus U) \otimes y \rightsquigarrow (S, T \oplus U) \otimes y \rightsquigarrow (0, yT) \rightsquigarrow (0, y f_* T)$$

– En passant par la droite on trouve :

$$(S, T \oplus U) \otimes y \rightsquigarrow (f_* S, f_* T + f_* U) \otimes y \rightsquigarrow (0, y f_* T)$$

2. $0 < \mathrm{d}^\circ(y) < q$

– gauche

$$(S, T \oplus U) \otimes y \rightsquigarrow (S, T \oplus U) \otimes f^* y \rightsquigarrow (T \wedge f^* y, 0) \rightsquigarrow \big(f_*(T \wedge f^* y), 0\big) = (f_* T \wedge y, 0)$$

– droite

$$(S, T \oplus U) \otimes y \rightsquigarrow (f_* S, f_* T \oplus f_* U) \otimes y \rightsquigarrow (f_* T \wedge y, 0)$$

3. $\mathrm{d}^\circ(y) = q$

– gauche

$$\begin{aligned}(S, T \oplus U) \otimes y &\rightsquigarrow (S, T \oplus U) \otimes f^* y \\ &\rightsquigarrow \big(T \wedge f^* y + S \wedge d(f^* y), (-1)^q U \wedge d(f^* y)\big) \\ &\rightsquigarrow \Big(f_*\big(T \wedge f^* y + S \wedge f^*(dy)\big), f_*\big((-1)^q U \wedge f^*(dy)\big)\Big) \\ &\qquad = \big(f_* T \wedge y + f_* S \wedge dy, (-1)^q U \wedge f^* \wedge dy\big)\end{aligned}$$

– droite

$$(S, T \oplus U) \otimes y \rightsquigarrow (f_* S, f_* T \oplus f_* U) \otimes dy \rightsquigarrow \big(f_* T \wedge y + f_* S \wedge dy, (-1)^q f_* U \wedge dy\big)$$

Ceci termine la preuve de (i).

Pour (ii), on peut tirer en arrière un courant T sur X par la submersion q en posant : $\langle q^* T, \alpha\rangle = \langle T, q_* \alpha\rangle$ où $q_* \alpha = \displaystyle\fint_q \alpha$ est l'intégrale de α sur la fibre de q. On a alors $q^* i_{Y*} T = i_{F*} p^* T$. $\square$

Bibliographie

[Bj] J.-E. Björk, Analytic $\mathcal{D}$-modules and Applications, Mathematics and its Applications, *247*, Kluwer Academic Publishers, (1993).

[Bl] S. Bloch, Semi-regularity and de Rham cohomology, Invent. Math., *17*, (1972), 51–66.

[EZZ] F. El Zein, S. Zucker, Extendability of normal functions associated to algebraic cycles, in *Topics in Transcendental Algebraic Geometry*, Annals of Math. Studies, *106*, Princeton University Press, (1984), 269–288.

[Es-Vi] H. Esnault, E. Viehweg, Deligne-Beilinson Cohomology, in *Beilinson's conjecture on special values of L-functions* (Rapoport, Schappacher & Schneider, eds.) Perspec. Math., Vol. 4, Academic Press, (1988), 43–91.

[Fe] H. Federer, Geometric Measure Theory, Springer, (1969).

[Gi-So] H. Gillet, C. Soulé, Arithmetic Chow groups and differential characters, in *Algebraic K-theory: Connections with Geometry and Topology,* Ed. J. F. Jardine, V. P. Snaith, Kluwer, (1989), 29–68.

[Gri-Ha] P. Griffiths, J. Harris, Principles of Algebraic Geometry, John Wiley and Sons, (1978).

[Gr] A. Grothendieck, Fondements de la géométrie algébrique, Secrétariat Mathématique, 11, rue Pierre Curie, Paris 5°, (1962).

[Ka] M. Kashiwara, $\mathcal{D}$-modules and microlocal calculus, Translation s of Mathematical Monographs, *217*, A.M.S., (2000).

[Ka-Sc] M. Kashiwara, P. Schapira, Sheaves on manifolds, Springer, (2002).

[Ki] J. King, The currents defined by analytic varieties, Acta Math., *127*, (1971), 185–220.

[Ko] H. Komatsu, Resolutions by hyperfunctions of sheaves of solutions of differential equations with constant coefficients, Math. Annalen, *176*, (1968), 77–86.

[Scha] P. Schapira, Théorie des hyperfonctions, Lecture Notes in Math. *126*, (1970), Springer.

[Sche] G. Scheja, Riemannsche Hebbarkeitssätze für Cohomologieklassen, Math. Ann. *144*, (1961), 345–360.

[Vo] C. Voisin, Théorie de Hodge et géométrie algébrique complexe, Société Mathématique de France, (2002).

CHAPITRE 4

Propriétés topologiques des schémas de Hilbert ponctuels des variétés symplectiques de dimension 4

Ce chapitre contient une version détaillée de l'article « Topological properties of punctual Hilbert schemes of symplectic 4-dimensional manifolds ».

Topological properties of punctual Hilbert schemes of symplectic 4-dimensional manifolds

Abstract. — In this article, we study topological properties of Voisin's almost-complex and symplectic punctual Hilbert schemes of a 4-manifold. We describe the ring structure and the cobordism class of these Hilbert schemes and we prove in this context a particular case of Ruan's crepant conjecture.

1. Introduction

Our aim in this paper is to extend some properties of punctual Hilbert schemes on smooth projective surfaces to the case of almost-complex or symplectic compact manifolds of dimension four.

Let X be a smooth complex projective surface. For any integer $n \in \mathbb{N}^*$, the punctual Hilbert scheme $X^{[n]}$ is defined as the set of all 0-dimensional subschemes of X of length n. A theorem of Fogarty [**Fo**] states that $X^{[n]}$ is a smooth irreducible projective variety of complex dimension $2n$ and that the Hilbert-Chow map $HC\colon X^{[n]} \longrightarrow S^nX$ defined by $HC(\xi) = \sum_{x\in\mathrm{supp}(\xi)} \ell_x(\xi)x$ is a desingularization of the symmetric product S^nX. This implies that the varieties $X^{[n]}$ can be seen as smooth compactifications of the sets of distinct unordered n-tuples of points in X. The cohomological and K-theoretical properties of these varieties $X^{[n]}$ have been intensively studied in the last few years: see for instance the references [**El-Gö-Le**], [**Fa-Gö**], [**Gö 2**], [**Gö 1**], [**Ja-Ka-Ki**], [**Le**], [**Le-So-2**], [**Li-Qi-Wa-1**], [**Na**], which provide main motivations for the present article.

It is proved in [**El-Gö-Le**] that the complex cobordism class of $X^{[n]}$ depends only on the complex cobordism class of X. This is what motivated Voisin to construct in [**Vo 1**] the Hilbert scheme $X^{[n]}$ when X is only supposed to be a smooth almost-complex compact fourfold. This construction of Voisin produces almost-complex Hilbert schemes $X^{[n]}$ which are differentiable manifolds of dimension $4n$ endowed with a stable almost-complex structure. Moreover there exists a continuous Hilbert-Chow map $HC\colon X^{[n]}\longrightarrow S^nX$ whose fibers are homeomorphic to the fibers of the Hilbert-Chow map in the integrable case. When X is furthermore assumed to be symplectic, $X^{[n]}$ is also symplectic, hence in particular almost-complex [**Vo 2**].

In this paper, using ideas of Voisin concerning relative integrable structures, we generalize to the almost-complex or symplectic setting many results already known in the integrable case, presenting at the same time an overview of the existing theory.

We begin by computing the Betti numbers of $X^{[n]}$, thus extending to the almost-complex case Göttsche's formula [**Gö 1**], [**Gö-So**].

Theorem 1.1. *Let (X,J) be an almost-complex compact fourfold and, for any positive integer n, let $\Big(b_i\big(X^{[n]}\big)\Big)_{i=0,\dots,4n}$ be the sequence of Betti numbers of the almost-complex Hilbert scheme $X^{[n]}$. Then the generating function for these Betti numbers is given by the formula*

$$\sum_{n=0}^{\infty}\sum_{i=0}^{4n} b_i\big(X^{[n]}\big)t^iq^n=\prod_{m=1}^{\infty}\frac{\big[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\big]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

The proof of Theorem 1.1 relies on a topological version for semi-small maps of the decomposition theorem for DGM sheaves of [**De-Be-Be-Ga**].

The second part of the paper is devoted to the definition and the study of the Nakajima operators $q_i(\alpha)$, $i\in\mathbb{Z}$, $\alpha\in H^*(X,\mathbb{Q})$ of an arbitrary almost-complex compact fourfold X. We prove in this context the Nakajima commutation relations:

Theorem 1.2. *For any pair (i,j) of integers and any pair (α,β) of cohomology classes in $H^*(X,\mathbb{Q})$ we have*

$$\big[q_i(\alpha),q_j(\beta)\big]=i\,\delta_{i+j,0}\int_X\alpha\beta.$$

The third part focuses on the computation of the boundary operator in the case where X is now supposed to be symplectic. The motivation for this study comes from the work of [**Le**]: this computation is the essential step in the understanding of the ring structure of $H^*\big(X^{[n]},\mathbb{Q}\big)$ in the classical theory. The boundary operator is the cup product operator with the class $-\frac{1}{2}\Big[\partial X^{[n]}\Big]$ in $H^2(X^{[n]},\mathbb{Q})$, where $\partial X^{[n]}\subseteq X^{[n]}$ consists of all the schemes which are not supported by n distinct points; this is also the exceptional divisor of the Hilbert-Chow map. In order to derive the formula for the boundary operator, we use the symplectic assumption via Donaldson's symplectic Kodaira theorem [**Do**] which provides many $\widetilde{J}$-holomorphic curves, where $\widetilde{J}$ is a suitable small perturbation of J. The main result of Part 4 is the following:

Theorem 1.3. *Let (X,ω) be a compact symplectic fourfold and J be a compatible almost-complex structure on X. If ∂ is the boundary operator, then for any pair of integers (n,m) and any pair (α,β) of classes in $H^*(X,\mathbb{Q})$,*

$$\big[\partial\,q_n(\alpha)-q_n(\alpha)\partial,q_m(\beta)\big]=-nm\left\{q_{n+m}(\alpha\beta)-\frac{|n|-1}{2}\,\delta_{n+m,0}\left(\int_X c_1(X)\alpha\beta\right)\mathrm{id}\right\}.$$

Part 5 deals with the ring structure of the almost-complex Hilbert schemes of a symplectic compact fourfold with vanishing first Betti number. In the algebraic case, Lehn's work [**Le**] describes completely the subring of $H^*\big(X^{[n]},\mathbb{Q}\big)$ generated by the Chern classes of $E^{[n]}$, where E is an arbitrary vector bundle on X and $E^{[n]}$ is the associated tautological bundle on $X^{[n]}$. These computations have been further developed in [**Li-Qi-Wa-1**], [**Li-Qi-Wa-3**], [**Le-So-2**] and [**Le-So-1**] in order to describe explicitely the ring $H^*\big(X^{[n]},\mathbb{Q}\big)$. Here is the main theorem of Part 5:

THEOREM 1.4. *Let (X,ω) be a compact symplectic fourfold with $b_1(X)=0$.*

(i) *The cohomology ring $H^*\big(X^{[n]},\mathbb{Q}\big)$ of the symplectic Hilbert scheme $X^{[n]}$ can be constructed by universal formulae starting with $H^*(X,\mathbb{Q})$, $c_1(X)$ and $c_2(X)$.*
(ii) *If $c_1(X)=0$ in $H^2(X,\mathbb{Q})$, $H^*\big(X^{[n]},\mathbb{Q}\big)$ is isomorphic to the orbifold cohomology ring $H^*_{CR}\big(S^nX,\mathbb{Q}\big)$ of the symmetric product S^nX.*

The first part of this theorem is due to Li, Qin and Wang [**Li-Qi-Wa-3**] in the algebraic case, see also [**Le-So-2**].

The assumption $b_1(X)=0$ in this theorem allows us to avoid the formalism of virtual Chern characters developed in [**Li-Qi-Wa-2**]. The second point of Theorem 4 solves Ruan's crepant resolution conjecture for symplectic symmetric products. In the algebraic case, X is a $K3$ surface or an Enriques surface, and Ruan's conjecture has been proved by Lehn and Sorger in [**Le-So-1**].

So far, we have concentrated on the cohomology ring on $X^{[n]}$. We study in Part 6 the class $\big[TX^{[n]}\big]$ in $K\big(X^{[n]}\big)$, where X is an arbitrary almost complex compact fourfold. We obtain in this context the already mentioned result of [**El-Gö-Le**]:

THEOREM 1.5. *Let (X,J) be an almost-complex compact fourfold and $X^{[n]}$, $n>0$, be the associated almost-complex Hilbert schemes. Then the almost-complex cobordism class of $X^{[n]}$ depends only on the almost-complex cobordism class of X.*

This theorem means that the function $X\longrightarrow X^{[n]}$ is defined at the level of cobordism and therefore fulfills the initial motivation of [**Vo 1**].

Acknowledgement. I want to thank my advisor Claire Voisin whose work on the almost-complex punctual Hilbert scheme is at the origin of this article. Her deep knowledge of the subject and her numerous advices have been most valuable to me. I also wish to thank her for her kindness and her patience.

2. The Hilbert schemes of an almost-complex compact fourfold

2.1. Definition and basic properties. In this section, we recall briefly Voisin's construction of the almost-complex Hilbert scheme and establish some complementary results. Let (X,J) be an almost-complex compact fourfold. The symmetric product S^nX will be endowed with the sheaf $\mathcal{C}^\infty_{S^nX}$ of $\mathcal{C}^\infty$ functions on X^n invariant by $\mathfrak{S}_n$. Let us introduce the incidence set

$$Z_n=\{(\underline{x},p)\in S^nX\times X,\ \text{such that}\ p\in\underline{x}\}. \tag{5}$$

If W is a small neighbourhood of Z_n in $S^nX\times X$, we consider the space $\mathcal{B}$ of relative integrable complex structures $J^{\,\mathrm{rel}}$ on the fibers of $pr_1\colon W\longrightarrow S^nX$ depending smoothly on the parameter $\underline{x}$ in S^nX and close to J in C^0 norm. The space $\mathcal{B}$ is contractible.

Let $\pi\colon (W^{[n]}_{\mathrm{rel}}, J^{\,\mathrm{rel}}) \longrightarrow S^nX$ be the relative Hilbert scheme of $(W, J^{\,\mathrm{rel}})$ over S^nX. The fibers of π are the smooth analytic sets $(W^{[n]}_{\underline{x}}, J^{\,\mathrm{rel}}_{\underline{x}})$, $\underline{x} \in S^nX$. Let $HC_{\mathrm{rel}}\colon W^{[n]}_{\mathrm{rel}} \longrightarrow S^n_{\mathrm{rel}}W$ be the relative Hilbert-Chow morphism.

Definition 2.1. The *topological Hilbert scheme* $(X^{[n]}, J^{\,\mathrm{rel}})$ is $(\pi, pr_2 \circ HC_{\mathrm{rel}})^{-1}\big(\Delta_{S^nX}\big)$, where Δ_{S^nX} is the diagonal of S^nX. More explicitely,

$$(X^{[n]}, J^{\,\mathrm{rel}}) = \{(\xi, \underline{x}) \text{ such that } \underline{x} \in S^nX,\ \xi \in (W^{[n]}_{\underline{x}}, J^{\,\mathrm{rel}}_{\underline{x}}) \text{ and } HC(\xi) = \underline{x}\,\}.$$

To put a differentiable structure on $(X^{[n]}, J^{\,\mathrm{rel}})$, Voisin uses specific relative integrable structures which are invariant by a compatible system of retractions on the strata of S^nX. These relative structures are differentiable for a differentiable structure $\mathfrak{D}_J$ on S^nX which depends on J and is weaker than the quotient differentiable structure, i.e. $\mathfrak{D}_J \subseteq \mathcal{C}^{\infty}_{S^nX}$. The main result of Voisin is the following:

Theorem 2.2. [**Vo 1**], [**Vo 2**]

(i) $X^{[n]}$ *is a $4n$-dimensional differentiable manifold, well-defined modulo diffeomorphisms homotopic to identity.*

(ii) *The Hilbert-Chow map $HC\colon X^{[n]} \longrightarrow \big(S^nX, \mathfrak{D}_J\big)$ is differentiable and its fibers $HC^{-1}(\underline{x})$ are homeomorphic to the fibers of the usual Hilbert-Chow morphism for any integrable structure near $\underline{x}$.*

(iii) *$X^{[n]}$ can be endowed with a stable almost-complex structure, hence defines an almost-complex cobordism class. When X is symplectic, $\big(X^{[n]}, J^{\,\mathrm{rel}}\big)$ is symplectic.*

The first point is the analogue of Fogarty's result [**Fo**] in the differentiable case. In our context, except in Part 6 where we will study the tangent bundle $TX^{[n]}$, we will not use differentiable properties of $X^{[n]}$ but only topological ones, which allows us to work with $\big(X^{[n]}, J^{\,\mathrm{rel}}\big)$ for any $J^{\,\mathrm{rel}}$ in $\mathcal{B}$. Without any assumption on $J^{\,\mathrm{rel}}$, the point (i) in Theorem 2.2 has the following topological version:

Proposition 2.3. *If $J^{\,\mathrm{rel}} \in \mathcal{B}$, $(X^{[n]}, J^{\,\mathrm{rel}})$ is a $4n$-dimensional topological manifold.*

Proof. Let $\underline{x}_0 \in S^nX$. There exist holomorphic relative coordinates $(z_{\underline{x}}, w_{\underline{x}})$ for $J^{\,\mathrm{rel}}_{\underline{x}}$ in a neighbourhood of $\underline{x}_0$ which depend smoothly on $\underline{x}$. For $\underline{x}$ near $\underline{x}_0$, the map $p \longrightarrow (z_{\underline{x}}(p), w_{\underline{x}}(p))$ is a biholomorphism between $(W_{\underline{x}}, J^{\,\mathrm{rel}}_{\underline{x}})$ and its image in $\mathbb{C}^2$ with the standard complex structure. Let us write $(z_{\underline{x}}(p), w_{\underline{x}}(p)) = \phi(\overline{x, p})$, where ϕ is a smooth function defined for x near a lift x_0 of $\underline{x}_0$, invariant by the action of the stabilizer of x_0 in $\mathfrak{S}_n$.
We write $x_0 = (y_1, \dots, y_1, \dots, y_k, \dots, y_k)$ where the points y_j are pairwise distinct and each y_j appears n_j times. We will identify small distinct neighbourhoods of y_j in X with distinct balls $B(y_j, \varepsilon)$ in $\mathbb{C}^2$. ϕ is defined on $B(y_1, \varepsilon)^{n_1} \times \dots \times B(y_k, \varepsilon)^{n_k} \times \cup_{j=1}^{k} B(y_j, \varepsilon)$ and is $\mathfrak{S}_{n_1} \times \dots \times \mathfrak{S}_{n_k}$ invariant. We can also suppose that $\phi(x_0, \,.\,) = \mathrm{id}$. We introduce new holomorphic coordinates by the formula $\widetilde{\phi}(x, p) = \phi(x, p) - D_1\,\phi(x_0, y_j)(x - x_0)$ if $p \in B(y_j, \varepsilon)$. Let Γ : $B(y_1, \varepsilon)^{n_1} \times \dots \times B(y_k, \varepsilon)^{n_k} \longrightarrow \big(\mathbb{C}^2\big)^n$ be defined by

$$\Gamma(x_1, \dots, x_n) = (\widetilde{\phi}(x_1, \dots, x_n, x_1), \dots, \widetilde{\phi}(x_1, \dots, x_n, x_n)).$$

The map Γ is $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$ equivariant and has identity differential at x_0, so it induces a local homeomorphism γ of S^nX around $\underline{x}_0$. The image of the chart of $(X^{[n]}, J^{\,\mathrm{rel}})$ above a neighbourhood of $\underline{x}_0$ will be the classical Hilbert scheme $(\mathbb{C}^2)^{[n]}$ above a neighbourhood of $\underline{x}_0$. The chart and its inverse are given by the formulae $\varphi(\xi) = \phi(\underline{x},\,.\,)_*$, where $HC(\xi) = \underline{x}$, and $\varphi^{-1}(\eta) = (\phi(\underline{y},\,.\,)^{-1})_*\eta$, where $\underline{y} = \gamma^{-1}(HC(\eta))$. □

Remark 2.4. Let J_0^{rel} and J_1^{rel} be two relative integrable complex structures, and let ϕ_0, ϕ_1, γ_0 and γ_1 be defined as above. Then $(X^{[n]}, J_0^{\mathrm{rel}})$ and $(X^{[n]}, J_1^{\mathrm{rel}})$ are homeomorphic above a neighbourhood of $\underline{x}_0$. If $\phi(\underline{x}, p) = \phi_1^{-1}(\gamma_1^{-1}\gamma_0(\underline{x}), \phi_0(\underline{x}, p))$ and $\gamma(\underline{x}) = \gamma_1^{-1}\gamma_0(\underline{x})$, then there is a commutative diagram

$$\begin{array}{ccccccc}
(X^{[n]}, J_0^{\mathrm{rel}}) & \hookleftarrow & HC^{-1}(V_{\underline{x}_0}) & \xrightarrow[\sim]{\phi_*} & HC^{-1}(\widetilde{V}_{\underline{x}_0}) & \hookrightarrow & (X^{[n]}, J_1^{\mathrm{rel}}) \\
\downarrow{\scriptstyle HC} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle HC} \\
S^nX & \supseteq & V_{\underline{x}_0} & \xrightarrow[\sim]{\gamma} & \widetilde{V}_{\underline{x}_0} & \subseteq & S^nX
\end{array}$$

and γ is a stratified isomorphism.

2.2. Computation of the Betti numbers. We will now turn our attention to the cohomology of $(X^{[n]}, J^{\,\mathrm{rel}})$. The first step is the computation of the Betti numbers of $X^{[n]}$. We recall the proof of Göttsche and Soergel ([**Gö-So**]), then we show how to adapt it in the non-integrable case.

Let X and Y be irreducible algebraic complex varieties and $f\colon Y \longrightarrow X$ be a proper morphism. We assume that X is stratified in such a way that f is a topological fibration above each stratum X_ν. We denote by d_ν the real dimension of the largest irreducible component of the fiber. If $Y_\nu = f^{-1}(X_\nu)$, $\mathcal{L}_\nu = R^{d_\nu} f_*\mathbb{Q}_{Y_\nu}$ will be the associated monodromy local system on X_ν.

Definition 2.5. – The map f is a *semi-small morphism* if for all ν, $\operatorname{codim}_X X_\nu \geq d_\nu$.
– A stratum X_ν is *essential* if $\operatorname{codim}_X X_\nu = d_\nu$.

We will say that Y is *rationally smooth* if the dualizing complex ω_Y with coefficients in $\mathbb{Q}$ is quasi-isomorphic to $\mathbb{Q}_Y[2\dim Y]$. This is the case if Y is smooth.

Theorem 2.6. [**De-Be-Be-Ga**] *If Y is rationally smooth and $f\colon Y \longrightarrow X$ is a proper semi-small morphism, there exists a canonical quasi-isomorphism*

$$Rf_*\mathbb{Q}_Y \xrightarrow{\sim} \bigoplus_{\nu \text{ essential}} j_{\nu *}\, IC_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu],$$

where $IC_{\overline{X}_\nu}(\mathcal{L}_\nu)$ is the intersection complex on $\overline{X}_\nu$ associated to the monodromy local system $\mathcal{L}_\nu$ and $j_\nu\colon \overline{X}_\nu \longrightarrow X$ is the inclusion. In particular, $H^k(Y, \mathbb{Q}) = \bigoplus_{\nu \text{ essential}} IH^{k-d_\nu}(\overline{X}_\nu, \mathcal{L}_\nu)$.

Remark 2.7. A simple proof of Theorem 2.6 is in [**LP**]. Furthermore, this proof shows that $Rf_*\mathbb{Q}_Y \simeq \bigoplus_{\nu \text{ essential}} j_{\nu *}\, IC_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu]$ under the following weaker topological hypotheses:
– Y is a rationally smooth topological space,
– X is a stratified topological space,
– $f\colon Y \longrightarrow X$ is a proper map which is locally equivalent on X to a semi-small map between complex analytic varieties.

Unfortunately, the notes [**LP**] have not been published. We provide Le Potier's proof of the decomposition theorem in Appendix II.

If X is a quasi-projective surface, the Hilbert-Chow morphism is semi-small with irreducible fibers, so that the monodromy local systems are trivial, and $X^{[n]}$ is smooth. The decomposition theorem gives Göttsche's formula for $b_i(X^{[n]})$. We now show that Göttsche's formula holds more generally for almost-complex Hilbert schemes.

THEOREM 2.8 (Göttsche's formula). *If (X, J) is an almost-complex compact fourfold, then for any integrable complex structure $J^{\,\mathrm{rel}}$,*

$$\sum_{n=0}^{\infty}\sum_{i=0}^{4n} b_i\big(X^{[n]}, J^{\,\mathrm{rel}}\big)t^i q^n = \prod_{m=1}^{\infty}\frac{\big[(1+t^{2m-1}q^m)(1+t^{2m+1}q^m)\big]^{b_1(X)}}{(1-t^{2m-2}q^m)(1-t^{2m+2}q^m)(1-t^{2m}q^m)^{b_2(X)}}.$$

PROOF. By Remark 2.7, it suffices to check that $HC\colon (X^{[n]}, J^{\,\mathrm{rel}}) \longrightarrow S^n X$ is locally equivalent to a semi-small morphism. The proof of Proposition 2.3 shows that $(X^{[n]}, J^{\,\mathrm{rel}}) \longrightarrow S^n X$ is locally equivalent to $HC\colon U^{[n]} \longrightarrow S^{[n]}U$ where U is an open set in $\mathbb{C}^2$. Thus the decomposition theorem applies and the computations are the same as in the integrable case. □

2.3. The homeomorphism type of almost-complex Hilbert schemes. Theorem 2.8 implies in particular that the Betti numbers of $(X^{[n]}, J^{\,\mathrm{rel}})$ are independent of $J^{\,\mathrm{rel}}$. We now prove a stronger result, namely that the Hilbert schemes corresponding to different relative integrable complex structures are homeomorphic.

PROPOSITION 2.9. *Let $\big(J_t^{\mathrm{rel}}\big)_{t\in B(0,r)\subseteq\mathbb{R}^n}$ be a family of smooth relative complex structures in a neighbourhood of Z_n varying smoothly with t, and $\varepsilon > 0$. Then there exists a continuous map $\psi\colon (t, \underline{x}, p) \longrightarrow \psi_{t,\underline{x}}(p)$ defined for $\|t\| < \varepsilon$, $(\underline{x}, p)$ in a neighbourhood of Z_n, with values in X, such that:*

(i) $\psi_{0,\underline{x}}(p) = p$,

(ii) *$\psi_{t,\underline{x}}$ is a biholomorphism between a neighbourhood of $\underline{x}$ and a neighbourhood of $S^n\psi_{t,\underline{x}}(\underline{x})$ with the structures $J^{\mathrm{rel}}_{0,\underline{x}}$ and $J^{\mathrm{rel}}_{t,\psi_{t,\underline{x}}(\underline{x})}$,*

(iii) *$\forall t \in B(0,\varepsilon)$, the map $\underline{x} \longrightarrow S^n\psi_{t,\underline{x}}(\underline{x})$ is a homeomorphism of $S^n X$.*

PROOF. We can choose a family of maps θ_t varying smoothly with t such that $\forall \underline{x} \in S^n X$ and $\forall t \in B(0, r)$, $\theta_{t,\underline{x}}$ is a biholomorphism between two neighbourhoods of $\underline{x}$ endowed with the structures $J^{\mathrm{rel}}_{t,\underline{x}}$ and $J^{\mathrm{rel}}_{0,\underline{x}}$, and such that $\forall \underline{x} \in S^n X$, $\theta_{0,\underline{x}} = \mathrm{id}$. We take, as in the proof of Proposition 2.3, a system $\big(\phi^i_{\underline{x}}\big)_{1\le i\le N}$ of holomorphic relative coordinates for J_0^{rel} with respect to a covering $\big(\widetilde{U}_i\big)_{1\le i\le N}$ of $S^n X$ such that $\underline{x} \longrightarrow S^n\phi^i_{\underline{x}}(\underline{x})$ is a homeomorphism between $\widetilde{U}_i$ and its image $\widetilde{V}_i$ in $S^n\mathbb{C}^2$. We define holomorphic relative coordinates $\big(\phi^i_{t,\underline{x}}\big)_{1\le i\le N}$ for J_t^{rel} by the formula $\phi^i_{t,\underline{x}}(p) = \phi^i_{\underline{x}}\big(\theta_{t,\underline{x}}(p)\big)$. For small t, after shrinking $\widetilde{U}_i$ if necessary, $\underline{x} \longrightarrow S^n\phi^i_{t,\underline{x}}(\underline{x})$ is still a homeomorphism: indeed the map $\underline{x} \longrightarrow S^n\phi^i_{t,\underline{x}}(\underline{x})$ is obtained from the $\mathfrak{S}_{n_1}\times\cdots\times\mathfrak{S}_{n_k}$ equivariant smooth map

$$(x_1, \dots, x_n) \longrightarrow \big(\phi_t(x_1, \dots, x_n, x_1), \dots, \phi_t(x_1, \dots, x_n, x_n)\big)\ .$$

Then we use the fact that a sufficiently small smooth perturbation of a smooth diffeomorphism remains a smooth diffeomorphism.

Let $\widetilde{Z}_n \subseteq S^n\mathbb{C}^2 \times \mathbb{C}^2$ be the incidence variety of $\mathbb{C}^2$. The map $\check{\phi}^i_t\colon (\underline{x}, p) \longrightarrow \big(S^n\phi^i_{t,\underline{x}}(\underline{x}), \phi^i_{t,\underline{x}}(p)\big)$ is a homeomorphism between two neighbourhoods of Z_n and $\widetilde{Z}_n$ above $\widetilde{U}_i$ and $\widetilde{V}_i$. If we define $\check{\psi}_t\colon (\underline{x}, p) \longrightarrow \big(S^n\psi_{t,\underline{x}}(\underline{x}), \psi_{t,\underline{x}}(p)\big)$, condition (ii) of the proposition means that $\check{\phi}^i_t \circ \check{\psi}_t \circ \big(\check{\phi}^i_0\big)^{-1}$ is of the form $(\underline{y}, p) \longrightarrow \big(S^n u_{t,\underline{y}}(\underline{y}), u_{t,\underline{y}}(p)\big)$ where $\underline{y} \in \widetilde{V}_i$ and $u_{t,\underline{y}}$ is a biholomorphism between a neighbourhood of $\underline{y}$ and its image (both endowed with the standard complex structure of $\mathbb{C}^2$), varying smoothly with t and $\underline{y}$. The condition (i) means that $u_{0,\underline{y}} = \mathrm{id}$. Thus $\big(\psi_t\big)_{||t|| \leq \varepsilon}$ can be constructed on small open sets of S^nX. Since biholomorphisms close to identity form a contractible set, we can, using cut-off functions, glue together the local solutions on S^nX to obtain a global one. The map $\underline{x} \longrightarrow S^n\psi_{t,\underline{x}}(\underline{x})$ is induced by a smooth $\mathfrak{S}_n$-equivariant map of X^n into X^n (and is a small perturbation of the identity map if $||t||$ is small enough), thus a $\mathfrak{S}_n$-equivariant diffeomorphism of X^n. We have therefore defined a family of maps $\big(\psi_t\big)_{||t|| \leq \varepsilon}$ satisfying the conditions (i), (ii) and (iii). □

Proposition 2.9 has important consequences:

Proposition 2.10. *Let $\big(J^{\mathrm{rel}}_t\big)_{t \in B(0,r)}$ be a smooth family of smooth relative complex structures. Then the relative Hilbert scheme $\big(X^{[n]}, \{J^{\mathrm{rel}}_t\}_{t \in B(0,r)}\big)$ is a topologically trivial fibration.*

Proof. We have

$$\big(X^{[n]}, \{J^{\mathrm{rel}}_t\}_{t \in B(0,r)}\big) = \Big\{(\xi, \underline{x}, t) \text{ such that } \underline{x} \in S^nX,\ t \in B(0,r),\ \xi \in \big(W^{[n]}_{\underline{x}}, J^{\mathrm{rel}}_{t,\underline{x}}\big)\Big\}.$$

A trivialization near zero is given by the map $\Gamma\colon \big(X^{[n]}, J^{\mathrm{rel}}_0\big) \times B(0,\varepsilon) \longrightarrow \big(X^{[n]}, \{J^{\mathrm{rel}}_t\}_{t \in B(0,\varepsilon)}\big)$ defined by $\Gamma(\xi, \underline{x}, t) = \big(\psi_{t,\underline{x}*}\xi, \psi_{t,\underline{x}}(\underline{x}), t\big)$. This proves that the relative Hilbert scheme is locally topologically trivial over $\overline{B}(0,r)$, hence globally trivial. □

Proposition 2.11. *If J^{rel}_0 and J^{rel}_1 are two relative integrable complex structures close to J, then $H^*\big(X^{[n]}, J^{\mathrm{rel}}_0, \mathbb{Q}\big)$ (resp. $K\big(X^{[n]}, J^{\mathrm{rel}}_0\big)$) and $H^*\big(X^{[n]}, J^{\mathrm{rel}}_1, \mathbb{Q}\big)$ (resp. $K\big(X^{[n]}, J^{\mathrm{rel}}_1\big)$) are canonically isomorphic.*

Proof. Proposition 2.10 shows that $(X^{[n]}, J^{\mathrm{rel}}_0)$ and $(X^{[n]}, J^{\mathrm{rel}}_1)$ are homeomorphic. If we consider two paths $\big(J^{\mathrm{rel}}_{0,t}\big)_{0 \leq t \leq 1}$ and $\big(J^{\mathrm{rel}}_{1,t}\big)_{0 \leq t \leq 1}$ between J^{rel}_0 and J^{rel}_1, since the set of relative integrable complex structures close to J is contractible, we can find a smooth family $\big(J^{\mathrm{rel}}_{s,t}\big)_{\substack{0 \leq s \leq 1 \\ 0 \leq t \leq 1}}$ which is an homotopy between the two paths. The relative associated Hilbert scheme over $[0,1] \times [0,1]$ is locally topologically trivial, hence globally trivial since $[0,1] \times [0,1]$ is contractible. This shows that the homeomorphisms between $(X^{[n]}, J^{\mathrm{rel}}_0)$ and $(X^{[n]}, J^{\mathrm{rel}}_1)$ constructed by the procedure above belong to a canonical homotopy class. □

As a consequence of this proposition, there exists a ring $H^*(X^{[n]}, \mathbb{Q})$ (resp. $K(X^{[n]})$) such that for any J^{rel} close to J, $H^*(X^{[n]}, \mathbb{Q})$ (resp. $K(X^{[n]})$) and $H^*(X^{[n]}, J^{\mathrm{rel}}, \mathbb{Q})$ (resp. $K(X^{[n]}, J^{\mathrm{rel}})$) are canonically isomorphic.

In the sequel, we will deal with products of Hilbert schemes. We can of course consider products of the type $\big(X^{[n]}, J^{\mathrm{rel}}_n\big) \times \big(X^{[m]}, J^{\mathrm{rel}}_m\big)$, but in practice it is necessary to work with pairs of relative integrable complex structures parametrized by $(\underline{x},\underline{y})$ in $S^nX\times S^mX$. Let us introduce the incidence set

$$Z_{n\times m}=\left\{(\underline{x},\underline{y},p)\text{ in }\left(S^nX\times S^mX\right)\times X\text{ such that }p\in\underline{x}\cup\underline{y}\right\}\cdot\tag{6}$$

Let W be a small neighbourhood of $Z_{n\times m}$ and let $J^{1,\mathrm{rel}}$ and $J^{2,\mathrm{rel}}$ be two relative integrable complex structures on the fibers of $pr_1\colon W\longrightarrow S^nX\times S^mX$.

Definition 2.12. The product Hilbert scheme $\left(X^{[n]\times[m]},J^{1,\mathrm{rel}},J^{2,\mathrm{rel}}\right)$ is defined by

$$\begin{aligned}\left(X^{[n]\times[m]},J^{1,\mathrm{rel}},J^{2,\mathrm{rel}}\right)=\Big\{(\xi,\eta,\underline{x},\underline{y})\text{ such that }\underline{x}\in S^nX,\ \underline{y}\in S^mX,\ \xi\in\left(W^{[n]}_{\underline{x},\underline{y}},J^{1,\mathrm{rel}}_{\underline{x},\underline{y}}\right),\\ \eta\in\left(W^{[m]}_{\underline{x},\underline{y}},J^{2,\mathrm{rel}}_{\underline{x},\underline{y}}\right),\ HC(\xi)=\underline{x},\ HC(\eta)=\underline{y}\Big\}\cdot\end{aligned}$$

The same definition goes for products of the type $\left(X^{[n_1]\times\cdots\times[n_k]},J^{\,1,\,\mathrm{rel}},\dots,J^{\,k,\,\mathrm{rel}}\right)$.

If there exist two relative integrable complex structures J_n^{rel} and J_m^{rel} in neighbourhoods of Z_n and Z_m such that $J^{1,\mathrm{rel}}_{\underline{x},\underline{y}}=J^{\mathrm{rel}}_{n,\underline{x}}$ and $J^{2,\mathrm{rel}}_{\underline{x},\underline{y}}=J^{\mathrm{rel}}_{m,\underline{x}}$ in small neighbourhoods of $\underline{x}$ and $\underline{y}$, we have

$$\left(X^{[n]\times[m]},J^{\,1,\mathrm{rel}},J^{\,2,\mathrm{rel}}\right)=\left(X^{[n]},J_n^{\,\mathrm{rel}}\right)\times\left(X^{[m]},J_m^{\,\mathrm{rel}}\right).$$

If $\left(J_t^{1,\mathrm{rel}},J_t^{2,\mathrm{rel}}\right)_{t\in B(0,r)}$ is a smooth family of relative integrable complex structures, then the family $\left(X^{[n]\times[m]},\left\{J_t^{1,\mathrm{rel}}\right\}_{t\in B(0,r)},\left\{J_t^{2,\mathrm{rel}}\right\}_{t\in B(0,r)}\right)$ is topologically trivial. A trivialization around 0 is given by

$$\begin{aligned}\left(X^{[n]\times[m]},J_0^{1,\mathrm{rel}},J_0^{2,\mathrm{rel}}\right)\times B(0,\varepsilon)&\longrightarrow\left(X^{[n]\times[m]},\left\{J_t^{1,\mathrm{rel}}\right\}_{t\in B(0,r)},\left\{J_t^{2,\mathrm{rel}}\right\}_{t\in B(0,r)}\right)\\ (\xi,\eta,\underline{x},\underline{y})\times t&\longmapsto\left(\psi_{t,\underline{x},\underline{y}*}\xi,\ \phi_{t,\underline{x},\underline{y}*}\eta,\ S^n\psi_{t,\underline{x},\underline{y}}(\underline{x}),\ S^m\phi_{t,\underline{x},\underline{y}}(\underline{y}),t\right)\end{aligned}$$

where

(i) For $p\in W_{\underline{x},\underline{y}}$ we have $\psi_{0,\underline{x},\underline{y}}(p)=\phi_{0,\underline{x},\underline{y}}(p)=p$.

(ii) The map $\psi_{t,\underline{x},\underline{y}}$ (resp. $\phi_{t,\underline{x},\underline{y}}$) is a biholomorphism between a neighbourhood of $\underline{x}\cup\underline{y}$ and a neighbourhood of $S^n\psi_{t,\underline{x},\underline{y}}(\underline{x})\cup S^m\psi_{t,\underline{x},\underline{y}}(\underline{y})$ (resp. $S^n\phi_{t,\underline{x},\underline{y}}(\underline{x})\cup S^m\phi_{t,\underline{x},\underline{y}}(\underline{y})$) with the structures $J^{1,\mathrm{rel}}_{0,\underline{x},\underline{y}}$ and $J^{1,\mathrm{rel}}_{t,S^n\psi_{t,\underline{x},\underline{y}}(\underline{x}),S^m\psi_{t,\underline{x},\underline{y}}(\underline{y})}$ (resp. $J^{2,\mathrm{rel}}_{0,\underline{x},\underline{y}}$ and $J^{2,\mathrm{rel}}_{t,S^n\phi_{t,\underline{x},\underline{y}}(\underline{x}),S^m\phi_{t,\underline{x},\underline{y}}(\underline{y})}$).

(iii) The map $(\underline{x},\underline{y})\longrightarrow\left(S^n\psi_{t,\underline{x},\underline{y}}(\underline{x}),S^m\psi_{t,\underline{x},\underline{y}}(\underline{y})\right)$ is a homeomorphism of $S^nX\times S^mX$.

3. Incidence varieties

3.1. Definitions and basic properties. If J is an integrable complex structure on X, the *incidence variety* $X^{[n',n]}$ is classically defined by $X^{[n',n]}=\left\{\xi\in X^{[n]},\ \xi'\in X^{[n']}\text{ such that }\xi\subseteq\xi'\right\}\cdot$ $X^{[n',n]}$ is never smooth unless $n'=n+1$. We have three maps $\lambda\colon X^{[n',n]}\longrightarrow X^{[n]}$, $\nu\colon X^{[n',n]}\longrightarrow X^{[n']}$ and $\rho\colon X^{[n',n]}\longrightarrow S^{n'-n}X$ given by $\lambda(\xi,\xi')=\xi$, $\nu(\xi,\xi')=\xi'$ and $\rho(\xi,\xi')=\mathrm{supp}\left(\mathcal{I}_\xi/\mathcal{I}_{\xi'}\right)$. Furthermore, (λ,ν) is injective.

If J is not integrable, we can define $X^{[n',n]}$ using the relative construction. Let $J^{\mathrm{rel}}_{n\times(n'-n)}$ be a relative integrable complex structure in a neighbourhood of $Z_{n\times(n'-n)}$ in $S^nX\times S^{n'-n}X\times X$ (see (6)).

DEFINITION 3.1. The incidence variety $\big(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\big)$ is defined by

$$\big(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\big)=\Big\{(\xi,\xi',\underline{x},\underline{y}) \text{ such that } \underline{x}\in S^nX,\ \underline{y}\in S^{n'-n}X,\ \xi\in\big(W^{[n]}_{\underline{x}}, J^{\text{rel}}_{n\times(n'-n),\,\underline{x},\,\underline{y}}\big),$$
$$\xi'\in\big(W^{[n']}_{\underline{x}\cup\underline{y}}, J^{\text{rel}}_{n\times(n'-n),\,\underline{x},\,\underline{y}}\big),\ \xi\subseteq\xi',\ HC(\xi)=\underline{x},\ \rho(\xi,\xi')=\underline{y}\Big\}.$$

Let $J^{\text{rel}}_{n\times n'}$ be a relative integrable complex structure in a neighbourhood of $Z_{n\times n'}$ such that for every $\underline{u}\in S^nX$ and $\underline{v}\in S^{n'-n}X$, $J^{\text{rel}}_{n\times n',\underline{u},\underline{u}\cup\underline{v}}=J^{\text{rel}}_{n\times(n'-n),\underline{u},\underline{v}}$. Then

$$\big(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\big)\subseteq\big(X^{[n]\times[n']}, J^{\text{rel}}_{n\times n'}, J^{\text{rel}}_{n\times n'}\big).$$

If $\big\{J^{\text{rel}}_{t,n\times n'}\big\}_{t\in B(0,r)}$ is a family of relative complex structures, we take a trivialization of $\big(X^{[n]\times[n']},\big\{J^{\text{rel}}_{t,n\times n'}\big\}_{t\in B(0,r)},\big\{J^{\text{rel}}_{t,n\times n'}\big\}_{t\in B(0,r)}\big)$. We can suppose $\psi_{t,\underline{x},\underline{y}}=\phi_{t,\underline{x},\underline{y}}$. If we define the relative integrable complex structure $J^{\text{rel}}_{t,n\times(n'-n)}$ as $J^{\text{rel}}_{t,n\times(n'-n),\underline{u},\underline{v}}=J^{\text{rel}}_{t,\,n\times n',\underline{u},\,\underline{u}\cup\underline{v}}$, then the subfamily $\big(X^{[n',n]},\big\{J^{\text{rel}}_{t,n\times(n'-n)}\big\}_{t\in B(0,r)}\big)$ is sent by this trivialization to the product $U^{[n',n]}\times B(0,\varepsilon)$, where U is an open set of $\mathbb{C}^2$. This means that the pair $\big(X^{[n',n]},X^{[n]\times[n']}\big)$ is topologically trivial when put in families. The morphism $\big(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\big)\longrightarrow S^nX\times S^{n'-n}X$ is locally equivalent above $S^nX\times S^{n'-n}X$ to the morphism $U^{[n',n]}\longrightarrow S^nU\times S^{n'-n}U$. This enables us to define a stratification on $X^{[n',n]}$ by patching together the analytic stratifications of a collection of $U_i^{[n',n]}$ (if Z is an analytic set, the analytic stratification of Z is given by $Z_0=Z_0^{\text{reg}}$, $Z_1=(Z\setminus Z_0)^{\text{reg}}$ and so on). In this way, $\big(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\big)$ becomes a stratified CW-complex such that for each stratum S, $\dim(\overline{S}\backslash S)\leq\dim S-2$. Let us introduce the following notations:

(i) The inverse image of the small diagonal of S^nX by $HC\colon(X^{[n]},J^{\text{rel}}_n)\longrightarrow S^nX$ will be denoted by $\big(X^{[n]}_0,J^{\text{rel}}_n\big)$.

(ii) The inverse image of the small diagonal of $S^{n'-n}X$ by $\rho\colon\big(X^{[n',n]}, J^{\text{rel}}_{n\times(n'-n)}\big)\longrightarrow S^{n'-n}X$ will be denoted by $\big(X^{[n',n]}_0, J^{\text{rel}}_{n\times(n'-n)}\big)$.

In the integrable case, $X^{[n',n]}_0$ is stratified by analytic sets $\big(Z_l\big)_{l\geq 0}$ defined by

$$Z_l=\big\{(\xi,\xi')\in X^{[n',n]}_0 \text{ such that if } x=\rho(\xi,\xi'),\ \ell_x(\xi)=l\big\};\tag{7}$$

Z_0 is irreducible of complex dimension $n'+n+1$, and all the other Z_l have smaller dimensions. By the same argument as above, this stratification also exists in the almost complex case. We prove the topological irreducibility of Z_0 in the following lemma:

LEMMA 3.2. *Let $\big[\,\overline{Z}_0\big]$ be the fundamental homology class of $\overline{Z}_0$. Then*

$$H_{2(n'+n+1)}\big(X^{[n',n]}_0, J^{\text{rel}}_{n\times(n'-n)},\mathbb{Z}\big)=\mathbb{Z}\,.\big[\,\overline{Z}_0\big].$$

PROOF. It is enough to prove that the Borel-Moore homology group $H^{\text{lf}}_{2(n'+n+1)}(Z_0,\mathbb{Z})$ is $\mathbb{Z}$, since all the remaining strata $\big(Z_l\big)_{l\geq 1}$ have dimensions smaller than $2(n'+n+1)-2$. Let

$$\widetilde{Z}_0=\Big\{(\xi,\eta,\underline{x},p) \text{ such that } \underline{x}\in S^nX,\ p\in X,\ \xi\in\big(W^{[n]}_{\underline{x},\,(n'-n)p}, J^{\text{rel}}_{n\times(n'-n),\,\underline{x},\,(n'-n)p}\big),\ HC(\xi)=\underline{x},$$
$$\eta\in\big(W^{[n'-n]}_{\underline{x},\,(n'-n)p}, J^{\text{rel}}_{n\times(n'-n),\,\underline{x},\,(n'-n)p}\big),\ HC(\eta)=(n'-n)p\Big\}.$$

There is a natural inclusion $Z_0 \hookrightarrow \widetilde{Z}_0$ given by $\big(\xi, \xi', \underline{x}, (n'-n)p\big) \longrightarrow (\xi, \xi'_{|p}, \underline{x}, p)$. Remark that $\widetilde{Z}_0$ is compact. Since $\dim(\widetilde{Z}_0 \backslash Z_0) \leq 4n + 2(n'-n-1) = 2(n'+n-1)$, it suffices to show that $H_{2(n'+n+1)}(\widetilde{Z}_0, \mathbb{Z}) = \mathbb{Z}$. $\widetilde{Z}_0$ is a product-type Hilbert scheme homeomorphic to $\big(X^{[n]}, J_n^{\mathrm{rel}}\big) \times \big(X_0^{[n'-n]}, J_{n'-n}^{\mathrm{rel}}\big)$. Since $\big(X_0^{[n'-n]}, J_{n'-n}^{\mathrm{rel}}\big)$ is by Briançon's theorem [**Br**] a topological fibration on X whose fiber is homeomorphic to an irreducible algebraic variety of complex dimension $n'-n-1$, we obtain the result. $\square$

3.2. Nakajima operators. We are now going to define the Nakajima operators $q_n(\alpha)$, which play an essential part in the study of the cohomology rings of the Hilbert schemes $X^{[n]}$. If $n' > n$, let us define

$$Q^{[n',n]} = \overline{Z_0} \subseteq \big(X^{[n] \times [n']}, J_{n \times n'}^{\mathrm{rel}}, J_{n \times n'}^{\mathrm{rel}}\big) \times X, \tag{8}$$

where the last map is given by the relative residual morphism $\rho(\xi, \xi') = \mathrm{supp}\big(\mathcal{I}_{\xi} / \mathcal{I}_{\xi'}\big)$ and Z_0 is defined by (7). Since the pair $\big(Q^{[n',n]}, X^{[n] \times [n']} \times X\big)$ is topologically trivial when $J_{n \times n'}^{\mathrm{rel}}$ varies, the cycle class $\big[Q^{[n',n]}\big] \in H_{2(n'+n+1)}\big(X^{[n]} \times X^{[n']} \times X, \mathbb{Z}\big)$ is independent of $J_{n \times n'}^{\mathrm{rel}}$.

Definition 3.3. Let $\alpha \in H^*(X, \mathbb{Q})$ and $j \in \mathbb{N}^*$. We define $q_j(\alpha)$ and $q_{-j}(\alpha)$ as follows:

$$\begin{array}{cccc} q_j(\alpha): & \bigoplus_n H^*(X^{[n]}, \mathbb{Q}) & \longrightarrow & \bigoplus_n H^*(X^{[n+j]}, \mathbb{Q}) \\ & \tau & \longmapsto & PD^{-1}\Big[pr_{2*}\big(\big[Q^{[n+j,n]}\big] \cap (pr_3^*\alpha \cup pr_1^*\tau)\big)\Big] \end{array}$$

$$\begin{array}{cccc} q_{-j}(\alpha): & \bigoplus_n H^*(X^{[n+j]}, \mathbb{Q}) & \longrightarrow & \bigoplus_n H^*(X^{[n]}, \mathbb{Q}) \\ & \tau & \longmapsto & PD^{-1}\Big[pr_{1*}\big(\big[Q^{[n+j,n]}\big] \cap (pr_3^*\alpha \cup pr_2^*\tau)\big)\Big] \end{array}$$

where pr_1, pr_2 and pr_3 are the projections from $X^{[n]} \times X^{[n+j]} \times X$ to each factor and PD is the Poincaré duality isomorphism between cohomology and homology. We also set $q_0(\alpha) = 0$

Remark 3.4. Let $|\alpha|$ be the degree of α, then $q_j(\alpha)$ maps $H^i(X^{[n]}, \mathbb{Q})$ to $H^{i+|\alpha|+2j-2}(X^{[n+j]}, \mathbb{Q})$.

We now prove the following extension to the almost-complex case of Nakajima's theorem [**Na**]:

Theorem 3.5. *For i, $j \in \mathbb{Z}$ and α, $\beta \in H^*(X, \mathbb{Q})$ we have*

$$q_i(\alpha) q_j(\beta) - (-1)^{|\alpha|\,|\beta|} q_j(\beta) q_i(\alpha) = i\, \delta_{i+j,0} \left(\int_X \alpha\beta \right) \mathrm{id}$$

.

Proof. We adapt Nakajima's proof to our situation. The most interesting case is the computation of $[q_{-i}(\alpha), q_j(\beta)]$ when i and j are positive. We introduce the classes $\big[P_{ij}\big]$, resp. $\big[Q_{ij}\big]$ in

$$H_*\big(X^{[n]}, \mathbb{Q}\big) \otimes H_*\big(X^{[n-i]}, \mathbb{Q}\big) \otimes H_*\big(X^{[n-i+j]}, \mathbb{Q}\big) \otimes H_*\big(X, \mathbb{Q}\big) \otimes H_*\big(X, \mathbb{Q}\big), \qquad \text{resp.}$$
$$H_*\big(X^{[n]}, \mathbb{Q}\big) \otimes H_*\big(X^{[n+j]}, \mathbb{Q}\big) \otimes H_*\big(X^{[n-i+j]}, \mathbb{Q}\big) \otimes H_*\big(X, \mathbb{Q}\big) \otimes H_*\big(X, \mathbb{Q}\big),$$

as follows:

$$\big[P_{ij}\big] := p_{13*}\Big[p_{124}^*\big[Q^{[n,n-i]}\big] \cdot p_{235}^*\big[Q^{[n-i+j,n-i]}\big]\Big], \qquad \text{resp.}$$
$$\big[Q_{ij}\big] := p_{13*}\Big[p_{124}^*\big[Q^{[n+j,n]}\big] \cdot p_{235}^*\big[Q^{[n+j,n-i+j]}\big]\Big],$$

where $Q^{[r,s]}$ is defined in (8). Then $q_j(\beta)q_{-i}(\alpha)$, resp. $q_{-i}(\alpha)q_j(\beta)$, is given by

$$\tau \longmapsto PD^{-1}\Big[pr_{3*}\big([P_{ij}] \cap \big(pr_5^*\beta \cup pr_4^*\alpha \cup pr_1^*\tau\big)\big)\Big], \qquad \text{resp.}$$

$$\tau \longmapsto PD^{-1}\Big[pr_{3*}\big([Q_{ij}] \cap \big(pr_5^*\alpha \cup pr_4^*\beta \cup pr_1^*\tau\big)\big)\Big].$$

First we deform all the relative integrable complex structures into a single one parametrized by $S^nX \times S^{n-i}X \times S^{n-i+j}X \times S^2X$. We take a relative integrable complex structure $J^{\mathrm{rel}}_{n\times(n-i)}$ on $S^nX \times S^{n-i}X \times X$ defined in a neighbourhood of $Z_{n\times(n-i)}$. Then we can find a relative integrable complex structure $\widetilde{J}^{0,\mathrm{rel}}_{n\times(n-i)\times(n-i+j)}$ in a neighbourhood of $Z_{n\times(n-i)\times(n-i+j)}$ such that

$$\forall \underline{x} \in S^nX,\ \forall \underline{y} \in S^{n-i}X,\ \forall \underline{z} \in S^{n-i+j}X, \quad \widetilde{J}^{0,\mathrm{rel}}_{n\times(n-i)\times(n-i+j),\underline{x},\underline{y},\underline{z}} = J^{\mathrm{rel}}_{n\times(n-i),\underline{x},\underline{y}}$$

in a neighbourhood of $\underline{x}\cup\underline{y}$. In the same way, if J^{rel}_{n-i+j} is a relative integrable complex structure in a neighbourhood of Z_{n-i+j}, we can find a structure $\widetilde{J}^{1,\mathrm{rel}}_{n\times(n-i)\times(n-i+j)}$ such that

$$\forall \underline{x} \in S^nX,\ \forall \underline{y} \in S^{n-i}X,\ \forall \underline{z} \in S^{n-i+j}X, \quad \widetilde{J}^{1,\mathrm{rel}}_{n\times(n-i)\times(n-i+j),\underline{x},\underline{y},\underline{z}} = J^{\mathrm{rel}}_{n-i+j,\underline{z}}$$

in a neighbourhood of $\underline{z}$. In this way, using product Hilbert schemes introduced in Definition 2.12, we obtain the equality between $\big(X^{[n]\times[n-i]}, J^{\mathrm{rel}}_{n\times(n-i)}, J^{\mathrm{rel}}_{n\times(n-i)}\big) \times \Big(X^{[n-i+j]}, J^{\mathrm{rel}}_{n-i+j}\Big)$ and $\Big(X^{[n]\times[n-i]\times[n-i+j]},\ \widetilde{J}^{0,\mathrm{rel}}_{n\times(n-i)\times(n-i+j)}, \widetilde{J}^{0,\mathrm{rel}}_{n\times(n-i)\times(n-i+j)}, \widetilde{J}^{1,\mathrm{rel}}_{n\times(n-i)\times(n-i+j)}\Big)$.

Let $\big(\widetilde{J}^{t,\mathrm{rel}}\big)_{0\le t\le 1}$ be a smooth path between $\widetilde{J}^{0,\mathrm{rel}}$ and $\widetilde{J}^{1,\mathrm{rel}}$. There exists a topological fibration $\mathfrak{X}$ over $[0,1]$ such that $\mathfrak{X}_t = \big(X^{[n]\times[n-i]\times[n-i+j]}, \widetilde{J}^{t,\mathrm{rel}}, \widetilde{J}^{t,\mathrm{rel}}, \widetilde{J}^{1,\mathrm{rel}}\big) \times X$. As already seen, the couple $\big(Q^{[n]\times[n-i]}, \mathfrak{X}\big)$ is topologically trivial over $[0,1]$. We can find a structure $\widetilde{J}^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ in a neighbourhood of $Z_{n\times(n-i)\times(n-i+j)\times 2}$ which satisfies:

$$\forall(\underline{x},\underline{y},\underline{z},\underline{w}) \in S^nX\times S^{n-i}X\times S^{n-i+j}X\times S^2X,\ \widetilde{J}^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2,\underline{x},\underline{y},\underline{z},\underline{w}} = \widetilde{J}^{1,\,\mathrm{rel}}_{n\times(n-i)\times(n-i+j),\underline{x},\underline{y},\underline{z}}$$

in a neighbourhood of $\underline{x}\cup\underline{y}\cup\underline{z}$. We can finally deform $\widetilde{J}^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ to any relative structure $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ in a neighbourhood of $Z_{n\times(n-i)\times(n-i+j)\times 2}$. Let

$$Y = \Big(X^{[n]\times[n-i]\times[n-i+j]\times[1]\times[1]}, J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2} \leftarrow 5 \text{ times}\Big)$$

be the product Hilbert scheme obtained by taking the same structure $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ five times (see Definition 2.12), where $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ is identified with its pullback by

$$\mu\colon S^nX \times S^{n-i}X \times S^{n-i+j}X \times X \times X \longrightarrow S^nX \times S^{n-i}X \times S^{n-i+j}X \times S^2X.$$

Then, via the canonical isomorphism,

$$H_*(Y,\mathbb{Q}) \simeq H_*\big(X^{[n]},\mathbb{Q}\big) \otimes H_*\big(X^{[n-i]},\mathbb{Q}\big) \otimes H_*\big(X^{[n-i+j]},\mathbb{Q}\big) \otimes H_*(X,\mathbb{Q}) \otimes H_*(X,\mathbb{Q}),$$

the class $p_{124}^*\big[Q^{[n,n-i]}\big]$ is the homology class of the cycle

$$A = \Big\{(\xi,\xi',\xi'',s,t) \text{ such that } \xi' \subseteq \xi \text{ and } \rho(\xi',\xi) = s\Big\}.$$

In the same way, $p_{235}^*\big[Q^{[n-i+j,n-i]}\big]$ is the homology class of the cycle

$$B = \Big\{(\xi,\xi',\xi'',s,t) \text{ such that } \xi' \subseteq \xi'' \text{ and } \rho(\xi',\xi'') = t\Big\}.$$

We now study the intersection of the cycles A and B. Let $p \in A \cap B$. We choose relative coordinates $\phi_{\underline{x},\underline{y},\underline{z},s,t}$ for $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ such that

$$(\underline{x},\underline{y},\underline{z},s,t) \longmapsto \big(S^n\phi_{\underline{x},\underline{y},\underline{z},s,t}(\underline{x}), S^{n-i}\phi_{\underline{x},\underline{y},\underline{z},s,t}(\underline{y}), S^{n-i+j}\phi_{\underline{x},\underline{y},\underline{z},s,t}(\underline{z}), \phi_{\underline{x},\underline{y},\underline{z},s,t}(s), \phi_{\underline{x},\underline{y},\underline{z},s,t}(t)\big)$$

is a local homeomorphism. The associated map given by

$$(\xi,\underline{x},\xi',\underline{y},\xi'',\underline{z},s,t) \longmapsto \big(\phi_{\underline{x},\underline{y},\underline{z},s,t*}\xi, \phi_{\underline{x},\underline{y},\underline{z},s,t*}\xi', S^{n-i+j}\phi_{\underline{x},\underline{y},\underline{z},s,t*}\xi'', \phi_{\underline{x},\underline{y},\underline{z},s,t}(s), \phi_{\underline{x},\underline{y},\underline{z},s,t}(t)\big)$$

is a homeomorphism from a neighbourhood of p to its image in $(\mathbb{C}^2)^{[n]}\times(\mathbb{C}^2)^{[n-i]}\times(\mathbb{C}^2)^{[n-i+j]}\times \mathbb{C}^2\times\mathbb{C}^2$ which maps A and B to the classical cycles $p_{124}^{-1}Q^{[n,n-i]}$ and $p_{235}^{-1}Q^{[n-i+j,n-i]}$. In the integrable case, we know that in the open set $\{s\neq t\}$, $p_{124}^{-1}Q^{[n,n-i]}$ and $p_{235}^{-1}Q^{[n-i+j,n-i]}$ intersect generically transversally. By the argument above, this is still true in our context. If $\big(A\cap B\big)_{s\neq t} = C_{ij}$, we can write $[A]\cdot[B] = \big[\,\overline{C_{ij}}\,\big] + \iota_* R$ where $\iota : Y_{\{s=t\}} \hookrightarrow Y$ is the natural injection and $R\in H_{2(2n-i+j+2)}\big(Y_{\{s=t\}},\mathbb{Q}\big)$. We can do the same in

$$Y' = \Big(X^{[n]\times[n+j]\times[n-i+j]\times[1]\times[1]}, J^{\mathrm{rel}}_{n\times(n+j)\times(n-i+j)\times 2} \leftarrow 5 \text{ times}\Big)$$

with the cycles A' and B' defined by

$$A' = \Big\{(\xi,\xi',\xi'',s,t) \text{ such that } \xi\subseteq\xi',\ \rho(\xi,\xi')=s\Big\}$$
$$B' = \Big\{(\xi,\xi',\xi'',s,t) \text{ such that } \xi''\subseteq\xi',\ \rho(\xi'',\xi')=t\Big\}.$$

We put $D_{ij} = \big(A'\cap B'\big)_{s\neq t}$. Then $[A']\cdot[B'] = \big[\,\overline{D_{ij}}\,\big] + \iota'_* R'$, where $\iota' : Y'_{\{s=t\}} \hookrightarrow Y'$ is the injection and $R'\in H_{2(2n-i+j+2)}\big(Y'_{\{s=t\}},\mathbb{Q}\big)$. The class R (resp. R') can be chosen supported in $A\cap B\cap Y_{\{s=t\}}$ (resp. in $A'\cap B'\cap Y'_{\{s=t\}}$).

The following lemma describes the situation outside the diagonal $\{s=t\}$.

LEMMA 3.6. $p_{1345*}\Big(\big[\,\overline{C_{ij}}\,\big]\cap\big(pr_5^*\beta\cup pr_4^*\alpha\big)\Big) = (-1)^{|\alpha|\,|\beta|}p_{1345*}\Big(\big[\,\overline{D_{ij}}\,\big]\cap\big(pr_5^*\alpha\cup pr_4^*\beta\big)\Big)$.

PROOF. Let us introduce the incidence varieties

$$T = \Big\{(\underline{x},\underline{y},\underline{z},s,t)\in S^nX\times S^{n-i}X\times S^{n-i+j}X\times X\times X \text{ such that } \underline{x}=\underline{y}+is,\ \ \underline{z}=\underline{y}+jt\Big\}$$
$$T' = \Big\{(\underline{x},\underline{y},\underline{z},s,t)\in S^nX\times S^{n+j}X\times S^{n-i+j}X\times X\times X \text{ such that } \underline{y}=\underline{x}+js=\underline{z}+it\Big\}$$

Let Ω, Ω' be two small neighbourhoods of T and T' and W a neighbourhood of $Z_{n\times(n-i+j)\times 2}$ such that if $(\underline{x},\underline{y},\underline{z},s,t)\in\Omega$ (resp. Ω'), $\underline{y}\in W_{\underline{x},\underline{z},s,t}$. Let $J^{\mathrm{rel}}_{n\times(n-i+j)\times 2}$ be a relative integrable complex structure on W. After shrinking Ω and Ω' if necessary, we can consider two relative structures $J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2}$ and $J^{\mathrm{rel}}_{n\times(n+j)\times(n-i+j)\times 2}$ such that

$$\forall(\underline{x},\underline{y},\underline{z},s,t)\in\Omega,\quad J^{\mathrm{rel}}_{n\times(n-i)\times(n-i+j)\times 2,\underline{x},\underline{y},\underline{z},s,t} = J^{\mathrm{rel}}_{n\times(n-i+j)\times 2,\underline{x},\underline{z},s,t},$$
$$\forall(\underline{x},\underline{y},\underline{z},s,t)\in\Omega',\quad J^{\mathrm{rel}}_{n\times(n+j)\times(n-i+j)\times 2,\underline{x},\underline{y},\underline{z},s,t} = J^{\mathrm{rel}}_{n\times(n-i+j)\times 2,\underline{x},\underline{z},s,t}.$$

Let U (resp. U') be the points of Y (resp. Y') lying above Ω (resp. Ω'). We define two maps u and v as follows:

$$u : U \longrightarrow \big(X^{[n]\times[n-i+j]\times[1]\times[1]}, J^{\mathrm{rel}}_{n\times(n-i+j)\times 2} \leftarrow 4 \text{ times}\big), \quad (\xi,\xi',\xi'',s,t) \longmapsto (\xi,\xi'',s,t),$$

$$v : U' \longrightarrow \big(X^{[n]\times[n-i+j]\times[1]\times[1]}, J^{\mathrm{rel}}_{n\times(n-i+j)\times 2} \leftarrow 4 \text{ times}\big), \quad (\xi,\xi',\xi'',s,t) \longmapsto (\xi,\xi'',s,t)$$

These maps can be extended to global ones which are in the homotopy class of p_{1345} after taking homeomorphisms between $X^{[n]\times[n-i]\times[n-i+j]\times[1]\times[1]}$, $X^{[n]\times[n+j]\times[n-i+j]\times[1]\times[1]}$, $X^{[n]\times[n-i+j]\times[1]\times[1]}$ and $X^{[n]}\times X^{[n-i]}\times X^{[n-i+j]}\times X^2$, $X^{[n]}\times X^{[n+j]}\times X^{[n-i+j]}\times X^2$, $X^{[n]}\times X^{[n-i+j]}\times X^2$. As in the integrable case, there is an isomorphism $\phi : C_{ij} \xrightarrow{\simeq} D_{ij}$ given as follows: if $(\xi,\xi',\xi'',s,t)\in C_{ij}$ with $HC(\xi') = \underline{y}$, $HC(\xi) = \underline{y}+is$, $HC(\xi'') = \underline{y}+jt$, we put $\phi(\xi,\xi',\xi'',s,t) = (\xi,\widetilde{\xi},\xi'',t,s)$ where $\widetilde{\xi}$ is defined by $\widetilde{\xi}_{|p} = \xi'_{|p}$ if $p\in\underline{y}$, $p\notin\{s,t\}$, $\widetilde{\xi}_{|s} = \xi_{|s}$ and $\widetilde{\xi}_{|t} = \xi''_{|t}$. All these schemes are considered for the structure $J^{\mathrm{rel}}_{n\times(n-i+j)\times 2,\underline{x},\underline{z},s\cup t}$. Let $\partial C_{ij} = \overline{C_{ij}}\backslash C_{ij}$, $\partial D_{ij} = \overline{D_{ij}}\backslash D_{ij}$ and $S = u(\partial C_{ij}) = v(\partial D_{ij})$. Let $\pi : Y' \longrightarrow Y'$ be defined by $\pi(\xi,\xi',\xi'',s,t) = (\xi,\xi',\xi'',t,s)$. We have the following diagram, where all the maps are proper:

$$\begin{tikzcd}[column sep=small] Y\backslash\partial C_{ij} \supseteq C_{ij} \arrow[rr, dash, "\phi", "\simeq"'] \arrow[dr, "u"'] & & D_{ij} \subseteq Y'\backslash\partial D_{ij} \arrow[dl, "v\circ\pi"] \\ & X^{[n]\times[n-i+j]\times[1]\times[1]}\backslash S & \end{tikzcd}.$$

Thus we obtain in $H^{\mathrm{lf}}_{2(2n-i+j+2)}\big(X^{[n]\times[n-i+j]\times[1]\times[1]}\backslash S,\mathbb{Q}\big)$ the equality

$$u_*\big([C_{ij}]\cap(pr_5^*\beta\cup pr_4^*\alpha)\big) = v_*\big([D_{ij}]\cap(pr_4^*\beta\cup pr_5^*\alpha)\big).$$

Since $\dim S \le 2(2n-i+j+2)-2$, we have

$$H^{\mathrm{lf}}_{2(2n-i+j+2)}\big(X^{[n]\times[n-i+j]\times[1]\times[1]}\backslash S,\mathbb{Q}\big) \simeq H_{2(2n-i+j+2)}\big(X^{[n]\times[n-i+j]\times[1]\times[1]},\mathbb{Q}\big),$$

and we get $p_{1345*}\Big(\big[\,\overline{C_{ij}}\,\big]\cap\big(pr_5^*\beta\cup pr_4^*\alpha\big)\Big) = (-1)^{|\alpha|\,|\beta|}p_{1345*}\Big(\big[\,\overline{D_{ij}}\,\big]\cap\big(pr_5^*\alpha\cup pr_4^*\beta\big)\Big).$ □

By this lemma, in $\big[q_{-i}(\alpha), q_j(\beta)\big]$, the terms coming from $\overline{C_{ij}}$ and $\overline{D_{ij}}$ cancel out. It remains to deal with the excess intersection components along the diagonals $Y_{\{s=t\}}$ and $Y'_{\{s=t\}}$. We introduce the locus

$$\Gamma = \Big\{\big(\xi,\,\underline{x},\xi'',\underline{z},\,s,\,t\big)\in X^{[n]\times[n-i+j]\times[1]\times[1]} \text{ such that } s=t,\ \xi_{|p} = \xi''_{|p} \text{ for } p\neq s$$
$$\text{and } HC(\xi'') = HC(\xi)+(j-i)s \text{ if } j\ge i,\ HC(\xi) = HC(\xi'')+(i-j)s \text{ if } j\le i\Big\}.$$

Γ contains $u(A\cap B)$ and $v(A'\cap B')$. As before, the dimension count can be done as in the integrable case: $\dim\Gamma < 2(2n-i+j+2)$ if $i\neq j$ and if $i=j$, Γ contains a $2(2n+2)$-dimensional component, namely $\Delta_{X^{[n]}}\times\Delta_X$. All other components have lower dimensions. Thus, if $i\neq j$, $p_{1345*}(\iota_*R) = 0$ and $p_{1345*}(\iota'_*R') = 0$ since they are supported in Γ and have degree $2(2n-i+j+2)$. If $i=j$, then $p_{1345*}(\iota_*R)$ and $p_{1345*}(\iota'_*R')$ are proportional to the fundamental class of $\Delta_{X^{[n]}}\times\Delta_X$. Now $p_{45*}\big(\big[\Delta_{X^{[n]}}\times\Delta_X\big]\cap\big(pr_4^*\alpha\cup pr_5^*\beta\big)\big) = \int_X \alpha\beta.\big[\Delta_{X^{[n]}}\big]$

and we obtain $\big[q_{-i}(\alpha), q_i(\beta)\big] = \mu \displaystyle\int_X \alpha\beta.\,\mathrm{id}$ where $\mu \in \mathbb{Q}$. The computation of the multiplicity μ is a local problem on X which is solved in [**Gr**], [**El-St**]. It turns out that $\mu = -i$. □

REMARK 3.7. The proof remains quite similar for $i > 0$, $j > 0$. There is no excess term in this case. Indeed, $Y = X^{[n]\times[n+i]\times[n+i+j]\times[1]\times[1]}$, $\Gamma = X^{[n+i+j,n]} \subseteq X^{[n]\times[n+i]\times[n+i+j]\times[1]\times[1]}$ and $\dim \Gamma = 2(2n+i+j+1) < 2(2n+i+j+2)$.

3.3. Representations of Heisenberg and Virasoro algebras. We recall in this section how to combine Nakajima's commutation relations and the Göttsche formula for the Betti numbers of Hilbert schemes to prove that the full cohomology algebra of the Hilbert schemes is a highest weight representation of the Heisenberg algebra of $H^*(X)$.

Let us recall that a super vector space over a field k is a vector space V over k together with a decomposition $V = V^+ \oplus V^-$. For $v \in V^{\pm}$, let $|v| = 0$ if $v \in V^+$ and $|v| = 1$ if $v \in V^-$. A bilinear form $<,>$ on V is *supersymmetric* if $\forall v, w \in V^{\pm}$, $< w, v >= (-1)^{|v|\,|w|} < v, w >$. If $(V, < . >)$ is a super vector space with a bilinear supersymmetric form, the Heisenberg algebra $\mathcal{H}(V)$ is an infinite-dimensional graded algebra defined by generators and relations as follows:

DEFINITION 3.8. The algebra $\mathcal{H}(V)$ is the quotient of the free algebra generated by the symbols c and $h_i(v)$ ($i \in \mathbb{Z}^*$, $v \in V$) by the ideal spanned by the relations

(i) $\forall i \in \mathbb{Z}^*, \forall \lambda, \mu \in k, \forall v, v' \in V$, $h_i(\lambda v + \mu v') = \lambda h_i(v) + \mu h_i(v')$.

(ii) $\forall i \in \mathbb{Z}^*, \forall v \in V$, $c\,.\,h_i(v) = h_i(v)\,.\,c$.

(iii) $\forall i, j \in \mathbb{Z}^*, \forall v, v' \in V^{\pm}$, $h_i(v)\,.\,h_j(v') - (-1)^{|v|\,|v'|} h_j(v')\,.\,h_i(v) = i\,\delta_{i+j,0} < v, v' > c$.

The Nakajima operators constructed in the previous section define a representation of the Heisenberg algebra $\mathcal{H}(H^*(V,\mathbb{Q}))$ in $\mathbb{H} := \bigoplus_{i,n} H^i(X^{[n]},\mathbb{Q})$, obtained by sending $h_i(\alpha)$ to $q_i(\alpha)$ and c to id.

PROPOSITION 3.9. [Na] $\mathbb{H}$ *is an irreducible* $\mathcal{H}(H^*(X,\mathbb{Q}))$*-module generated by the vector 1.*

PROOF. Let $V = H^*(X,\mathbb{Q})$ and $W_+ = V \otimes t\,\mathbb{Q}[t]$. If $\mathcal{I}$ is the left ideal of $\mathcal{H}(V)$ generated by the $h_{-i}(v)$ ($v \in V$, $i \in \mathbb{N}^*$), $\mathcal{H}(V)\,/_{\mathcal{I}}$ is an irreducible $\mathcal{H}(V)$-module which turns out to be isomorphic to the supersymmetric algebra S^*W_+ of W_+. We consider the morphism of $\mathcal{H}$-modules $\mathcal{H} \longrightarrow \mathbb{H}$ given by $h_{n_1}(\alpha_1)\dots h_{n_k}(\alpha_k) \longrightarrow q_{n_1}(\alpha_1)\dots q_{n_k}(\alpha_k).1$. It induces a morphism $S^*W_+ \longrightarrow \mathbb{H}$. By Schur's lemma, it is either 0, which is impossible, or injective. Furthermore, the graded pieces $\big(S^*W_+\big)_n$ in $\mathbb{H}_n = H^*(X^{[n]},\mathbb{Q})$ have the same dimensions since, by Göttsche's formula,

$$\sum_{n=0}^{\infty} \dim\big(S^*W_+\big)_n\, q^n = \sum_{n=0}^{\infty} \dim H^*(X^{[n]},\mathbb{Q})\, q^n = \prod_{m=1}^{\infty} \frac{\big(1+q^m\big)^{2b_1(X)}}{\big(1-q^m\big)^{2+b_2(X)}}.$$

This implies $S^*W_+ \simeq \mathbb{H}$, and $\mathbb{H}$ becomes an irreducible $\mathcal{H}\big(H^*(X,\mathbb{Q})\big)$-module with highest weight vector 1. □

We define now the Virasoro operators. They are formally deduced from Nakajima's ones and will play an important part in the sequel.

First, we introduce some notations:

(i) If $p \in \mathbb{N}^*$, δ_p is the diagonal inclusion $X \hookrightarrow X^p$ and $\tau_p \colon H^*(X,\mathbb{Q}) \longrightarrow H^*(X,\mathbb{Q})^{\otimes p}$ is the Gysin morphism δ_{p*} combined with the Künneth isomorphism.

(ii) If $n_1, \dots, n_k \in \mathbb{Z}$, the generalized Nakajima operator $q_{n_1} \dots q_{n_k} \colon H^*(X, \mathbb{Q})^{\otimes k} \longrightarrow \mathrm{End}(\mathbb{H})$ is defined by $q_{n_1} \dots q_{n_k}(\alpha_1 \otimes \cdots \otimes \alpha_k) = q_{n_1}(\alpha_1) \circ \cdots \circ q_{n_k}(\alpha_k)$.

(iii) We use the physicists' normal ordering convention $: q_{n_1} \dots q_{n_k} : = q_{n_{i_1}} \dots q_{n_{i_k}}$ where $\{1, \dots, k\} \longrightarrow \{i_1, \dots, i_k\}$ is a permutation such that $n_{i_1} \geq \cdots \geq n_{i_k}$.

Definition 3.10. For $\alpha \in H^*(X, \mathbb{Q})$, the *Virasoro operators* $\big(\mathcal{L}_n(\alpha)\big)_{n \in \mathbb{Z}}$ are the elements of $\mathrm{End}(\mathbb{H})$ defined by $\mathcal{L}_n(\alpha) = \frac{1}{2} \sum\limits_{\nu \in \mathbb{Z}} : q_\nu q_{n-\nu} : \tau_2(\alpha)$.

Proposition 3.11.

(i) $\big[\mathcal{L}_n(\alpha), q_m(\beta)\big] = -m \, q_{n+m}(\alpha\beta)$.

(ii) $\big[\mathcal{L}_n(\alpha), \mathcal{L}_m(\beta)\big] = (n-m)\mathcal{L}_{n+m}(\alpha\beta) - \dfrac{n^3 - n}{12} \delta_{n+m,0} \left(\displaystyle\int_X c_2(X)\alpha\beta\right) \mathrm{id}$.

Proof. This is a consequence of Nakajima's relations. For the detailed computations see [**Le**], Theorem 3.3. $\square$

4. The boundary operator

4.1. Tautological bundles. We begin with preliminary material about tautological bundles on Hilbert schemes. In the classical case, let E be a algebraic vector bundle on an algebraic surface X. For $n \in \mathbb{N}$, we consider the incidence diagram

$$\begin{tikzcd} & X^{[n]} \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X^{[n]} & & X \end{tikzcd}$$

Let $Y_n \subseteq X^{[n]} \times X$ be the incidence locus, then $p_{\,|\,Y_n} \colon Y_n \longrightarrow X^{[n]}$ is finite. The tautological vector bundle $E^{[n]}$ is defined by $E^{[n]} = p_*\big(q^*E \otimes \mathcal{O}_{Y_n}\big)$. For every $\xi \in X^{[n]}$, $E^{[n]}_{|\,\xi} = H^0\big(\xi, i_\xi^* E\big)$. Our first aim is to generalize this construction in the almost-complex case.

Let (X, J) be an almost-complex compact fourfold, $Z_n \subseteq S^n X \times X$ the incidence locus, W a small neighbourhood of Z_n and J_n^{rel} a relative integrable structure on W. The fibers of $pr_1 \colon W \longrightarrow S^n X$ are smooth analytic sets. We endow W with the sheaf $\mathcal{A}_W$ of continuous functions which are smooth on the fibers of pr_1. We can consider the vector bundle $\mathcal{A}^{0,1}_{W,\mathrm{rel}}$ of relative $(0,1)$-forms on W, which satisfies $\forall \underline{x} \in S^n X$, $\mathcal{A}^{0,1}_{W,\mathrm{rel}\,|W_{\underline{x}}} = \mathcal{A}^{0,1}_{W_{\underline{x}}, J^{\mathrm{rel}}_{n,\underline{x}}}$. We have a relative $\overline{\partial}$-operator $\overline{\partial}^{\,\mathrm{rel}} \colon \mathcal{A}_W \longrightarrow \mathcal{A}^{0,1}_{W,\mathrm{rel}}$ which induces for each $\underline{x} \in S^n X$ the usual operator $\overline{\partial} \colon \mathcal{A}_{W_{\underline{x}}} \longrightarrow \mathcal{A}^{0,1}_{W_{\underline{x}}}$ given by the complex structure $J^{\mathrm{rel}}_{n,\underline{x}}$ on $W_{\underline{x}}$.

Definition 4.1. Let E be a complex vector bundle on X.

(i) A *relative connection* $\overline{\partial}_E^{\mathrm{rel}}$ *on* E *compatible with* J_n^{rel} is a $\mathbb{C}$-linear morphism of sheaves $\overline{\partial}_E : \mathcal{A}_W\big(pr_2^* E\big) \longrightarrow \mathcal{A}_W^{0,1}\big(pr_2^* E\big)$ satisfying $\overline{\partial}_E^{\mathrm{rel}}(\varphi s) = \varphi \, \overline{\partial}_E^{\mathrm{rel}} s + \overline{\partial}^{\,\mathrm{rel}} \varphi \otimes s$ for all sections φ and s of $\mathcal{A}_W$ and $\mathcal{A}_W\big(pr_2^* E\big)$ respectively.

(ii) A relative connection $\overline{\partial}_E^{\mathrm{rel}}$ is *integrable* if $\big(\overline{\partial}_E^{\mathrm{rel}}\big)^2 = 0$.

If $\overline{\partial}_E^{\mathrm{rel}}$ is an integrable connection on E compatible with J_n^{rel}, we can apply the Kozsul-Malgrange integrability theorem with continuous parameters in $S^n X$ [**Vo 3**]. Thus, for every $\underline{x} \in S^n X$, $E_{|W_{\underline{x}}}$

is endowed with the structure of a holomorphic vector bundle over $\left(W_{\underline{x}}, J_{n,\underline{x}}^{\mathrm{rel}}\right)$ and this structure varies continuously with $\underline{x}$. Furthermore, $\ker \overline{\partial}_{E}^{\mathrm{rel}}$ is the sheaf of relative holomorphic sections of E. Therefore, there is no difference between relative integrable connections on E compatible with J_n^{rel} and relative holomorphic structures on E compatible with J_n^{rel}.

Taking relative holomorphic coordinates for J_n^{rel}, we see that relative integrable connections exist on W above small open sets of S^nX. By a partition of unity on S^nX, we can build global ones. The space of holomorphic structures on a complex vector bundle above a ball in $\mathbb{C}^2$ is contractible. Therefore, the space of relative holomorphic structures on E compatible with J_n^{rel} is also contractible.

We proceed now to the construction of the tautological vector bundle $E^{[n]}$ on $\left(X^{[n]}, J_n^{\mathrm{rel}}\right)$. Let $\overline{\partial}_{E}^{\mathrm{rel}}$ be a relative holomorphic structure on E adapted to J_n^{rel}. We can construct a vector bundle $E_{\mathrm{rel}}^{[n]}$ over $W_{\mathrm{rel}}^{[n]}$ such that for every $\underline{x}$ in S^nX, $E^{[n]}_{\mathrm{rel}\,|W_{\underline{x}}^{[n]}} = E^{[n]}_{|W_{\underline{x}}^{[n]}}$, where $E_{|W_{\underline{x}}}$ is endowed with the holomorphic structure given by $\overline{\partial}_{E,\underline{x}}^{\mathrm{rel}}$. For this, we take holomorphic relative coordinates for E: if U is a small open set of S^nX, they are trivializations $z_1(\underline{x}), \dots, z_r(\underline{x})$ of $E_{|W_{\underline{x}}}$ for $\underline{x} \in U$ which satisfy $\overline{\partial}_{E,\underline{x}}^{\mathrm{rel}}(z_i(\underline{x})) = 0$ for $1 \leq i \leq r$. We can also take relative coordinates $(w_1,\, w_2)$ for the structure J_n^{rel} on the base: for $\underline{x} \in U$, $w_1(\underline{x})$ and $w_2(\underline{x})$ are holomorphic coordinates on $\left(W_{\underline{x}}, J_{n,\underline{x}}^{\mathrm{rel}}\right)$. In this way we obtain a diagram

$$\begin{tikzcd} \coprod_{\underline{x}\in U} E_{|W_{\underline{x}}} \arrow[r, hook, "\Gamma"] \arrow[d] & U \times \mathbb{T}_r \arrow[d] \\ \coprod_{\underline{x}\in U} W_{\underline{x}} \arrow[r, hook, "\psi"'] & U \times \mathbb{C}^2 \end{tikzcd}$$

where $\mathbb{T}_r = \mathbb{C}^2 \times \mathbb{C}^r$ is the trivial complex bundle of rank r on $\mathbb{C}^2$, $\psi(\underline{x}, p) = (\underline{x}, w_1(\underline{x})(p), w_2(\underline{x})(p))$ and $\Gamma^{-1}(\psi(\underline{x}, p), \lambda_1, \dots, \lambda_r) = \lambda_1 z_1(\underline{x})(p) + \dots + \lambda_r z_r(\underline{x})(p)$. The map Γ is relatively holomorphic over S^nX. Then we get another diagram

$$\begin{tikzcd} \coprod_{\underline{x}\in U} E^{[n]}_{|W_{\underline{x}}^{[n]}} \arrow[r, hook, "\Gamma^{[n]}"] \arrow[d] & U \times \mathbb{T}_r^{[n]} \arrow[d] \\ \coprod_{\underline{x}\in U} W_{\underline{x}}^{[n]} \arrow[r, hook, "\psi^{[n]}"'] & U \times (\mathbb{C}^2)^{[n]} \end{tikzcd}$$

The vector bundle $E_{\mathrm{rel}}^{[n]}$ is defined as a set by $E_{\mathrm{rel}}^{[n]} = \coprod_{\underline{x}\in S^nX} E^{[n]}_{|W_{\underline{x}}^{[n]}}$. A local chart for $E_{\mathrm{rel}}^{[n]}$ over S^nX is given by $\Gamma^{[n]}$.

Definition 4.2. Let $i\colon \left(X^{[n]}, J_n^{\mathrm{rel}}\right) \longrightarrow W_{\mathrm{rel}}^{[n]}$ be the canonical injection. The complex vector bundle $\left(E^{[n]}, J_n^{\mathrm{rel}}, \overline{\partial}_{E}^{\mathrm{rel}}\right)$ on $\left(X^{[n]}, J_n^{\mathrm{rel}}\right)$ is defined by $E^{[n]} = i^* E_{\mathrm{rel}}^{[n]}$.

In the sequel, we consider the class of $E^{[n]}$ in $K\big(X^{[n]}\big)$, which we prove below to be independent of the structures used in the construction.

PROPOSITION 4.3. *The class of $E^{[n]}$ in $K\big(X^{[n]}\big)$ is independent of $\big(J_n^{\mathrm{rel}}, \overline{\partial}_E^{\mathrm{rel}}\big)$.*

PROOF. Let $\big(J_{0,n}^{\mathrm{rel}}, \overline{\partial}_{E,0}^{\mathrm{rel}}\big)$ and $\big(J_{1,n}^{\mathrm{rel}}, \overline{\partial}_{E,1}^{\mathrm{rel}}\big)$ be two relative holomorphic structures and let $\big(J_{t,n}^{\mathrm{rel}}, \overline{\partial}_{E,t}^{\mathrm{rel}}\big)$ be a smooth path between them. Let $W_{\mathrm{rel}}^{[n]}$ be the relative Hilbert scheme over $S^nX \times [0,1]$ for the family $\big(J_{t,n}^{\mathrm{rel}}\big)_{0\leq t\leq 1}$. There exists a vector bundle $\big(\widetilde{E}_{\mathrm{rel}}^{[n]}, \{J_{t,n}^{\mathrm{rel}}\}_{0\leq t\leq 1}, \{\overline{\partial}_{E,t}^{\mathrm{rel}}\}_{0\leq t\leq 1}\big)$ over $W_{\mathrm{rel}}^{[n]}$ such that $\forall t\in[0,1]$, $\widetilde{E}^{[n]}_{\mathrm{rel}\,|\,W^{[n]}_{\mathrm{rel},t}} = \big(E_{\mathrm{rel}}^{[n]}, J_{t,n}^{\mathrm{rel}}, \overline{\partial}_{E,t}^{\mathrm{rel}}\big)$. If $\mathfrak{X} = \big(X^{[n]}, \{J_{t,n}^{\mathrm{rel}}\}_{0\leq t\leq 1}\big) \subseteq W_{\mathrm{rel}}^{[n]}$ is the relative Hilbert scheme over $[0,1]$, $\widetilde{E}^{[n]}_{\mathrm{rel}\,|\,\mathfrak{X}}$ is a complex vector bundle on $\mathfrak{X}$ whose restriction to $\mathfrak{X}_t$ is $\big(E^{[n]}, J_{t,n}^{\mathrm{rel}}, \overline{\partial}_{E,t}^{\mathrm{rel}}\big)$. Now $\mathfrak{X}$ is topologically trivial over $[0,1]$ by Proposition 2.10. Since $K(\mathfrak{X}_0\times[0,1]) \simeq K(\mathfrak{X}_0)$, we get the result. $\square$

Let $\mathbb{T} = X\times\mathbb{C}$ be the trivial complex line bundle on X and $\partial X^{[n]} \subseteq X^{[n]}$ be the inverse image of the big diagonal of S^nX by the Hilbert-Chow morphism. We have $\dim \partial X^{[n]} = 4n-2$ and $H_{4n-2}(\partial X^{[n]},\mathbb{Z}) \simeq \mathbb{Z}$ (this can be proved as in Lemma 3.2).

LEMMA 4.4. $c_1(\mathbb{T}^{[n]}) = -\frac{1}{2}\, PD^{-1}\big(\big[\partial X^{[n]}\big]\big)$.

PROOF. The line bundle $\mathbb{T}^{[n]}$ is canonically trivial above $X^{[n]}\backslash \partial X^{[n]}$; this implies that $c_1(\mathbb{T}^{[n]}) = \mu\, PD^{-1}\big([\partial X^{[n]}]\big)$ where $\mu\in\mathbb{Q}$. To compute μ, we argue locally on S^nX around a point in the stratum $S = \big\{\underline{x}\in S^nX,$ such that $x_i\neq x_j$ except for one pair $\{i,j\}\big\}$. This reduces the computation to the case $n=2$. Then $U^{[2]} = Bl_\Delta(U\times U)/\mathbb{Z}_2$, where $U\subseteq X$ is endowed with an integrable complex structure and Δ is the diagonal of U. If $E\subseteq Bl_\Delta(U\times U)$ is the exceptional divisor and $\pi\colon Bl_\Delta(U\times U)\longrightarrow U^{[2]}$ is the projection, then $\pi^*\big([\partial U^{[2]}]\big) = 2[E]$ and $\pi^*c_1(\mathbb{T}^{[2]}) = c_1(\pi^*\mathbb{T}^{[2]}) = c_1\big(\mathcal{O}(-E)\big) = -[E]$ in $H^2\big(Bl_\Delta(U\times U),\mathbb{Z}\big)$. This gives $\mu = -1/2$. $\square$

We want to compare the universal bundles $E^{[n]}$ and $E^{[n+1]}$ through the incidence variety $X^{[n+1,n]}$. In the integrable case, $X^{[n+1,n]}$ is smooth. If $D\subseteq X^{[n+1,n]}$ is the divisor $\overline{Z}_1$ (see (7)), then we have an exact sequence [**El-Gö-Le**], [**Le**], [**Da**]:

$$0\longrightarrow \rho^*E\otimes\mathcal{O}_{X^{[n+1,n]}}(-D)\longrightarrow \nu^*E^{[n+1]}\longrightarrow\lambda^*E^{[n]}\longrightarrow 0, \tag{9}$$

where $\lambda\colon X^{[n+1,n]}\longrightarrow X^{[n]}$, $\nu\colon X^{[n+1,n]}\longrightarrow X^{[n+1]}$ and $\rho\colon X^{[n+1,n]}\longrightarrow X$ are the two natural projections and the residual map.

In the almost-complex case, $X^{[n+1,n]}$ is a topological manifold of dimension $4n+4$. If we choose a relative integrable structure J_{n+1}^{rel} with additional properties as given in [**Vo 1**], $X^{[n+1,\,n]}$ can be endowed with a differentiable structure, but we will not need it here.

Let J_n^{rel} and J_{n+1}^{rel} be two relative integrable structures in small neighbourhoods of Z_n and Z_{n+1}. We extend them to relative structures $\check{J}_n^{\mathrm{rel}}$ and $\check{J}_{n+1}^{\mathrm{rel}}$ in small neighbourhoods of $Z_{n\times(n+1)}$. Then $\big(X^{[n]\times[n+1]}, \check{J}_n^{\mathrm{rel}}, \check{J}_{n+1}^{\mathrm{rel}}\big) = \big(X^{[n]}, J_n^{\mathrm{rel}}\big)\times\big(X^{[n+1]}, J_{n+1}^{\mathrm{rel}}\big)$. If $J_{n\times(n+1)}^{\mathrm{rel}}$ is a relative integrable structure in a small neighbourhood of $Z_{n\times(n+1)}$ and $J_{n\times 1}^{\mathrm{rel}}$ is defined by $J_{n\times 1,\underline{x},p}^{\mathrm{rel}} = J_{n\times(n+1),\,\underline{x},\,\underline{x}\cup p}^{\mathrm{rel}}$, then we have a diagram:

$$\begin{tikzcd}
 & & \left(X^{[n+1]}, J^{\mathrm{rel}}_{n+1}\right) \\
\left(X^{[n+1,n]}, J^{\mathrm{rel}}_{n\times 1}\right) \arrow[r, hook] \arrow[urr, bend left, "\nu"] \arrow[drr, bend right, "\lambda"'] & \left(X^{[n]\times[n+1]}, J^{\mathrm{rel}}_{n\times(n+1)}, J^{\mathrm{rel}}_{n\times(n+1)}\right) \arrow[r, "\Phi", "\simeq"'] & \left(X^{[n]\times[n+1]}, \check{J}^{\mathrm{rel}}_{n}, \check{J}^{\mathrm{rel}}_{n+1}\right) \arrow[u, "pr_1"'] \arrow[d, "pr_2"] \\
 & & \left(X^{[n]}, J^{\mathrm{rel}}_{n}\right)
\end{tikzcd}$$

where ϕ is a homeomorphism uniquely determined up to homotopy. We will denote by D the inverse image of the incidence locus of $S^nX \times X$ by the map $X^{[n+1,n]} \longrightarrow S^nX \times X$ so that $D = \overline{Z}_1$, where Z_1 is defined by (7). The cycle D has a fundamental homology class in $H_{4n+2}\big(X^{[n+1,n]}, \mathbb{Z}\big)$. There exists a unique complex line bundle F on $X^{[n+1,n]}$ such that $PD\big(c_1(F)\big) = -[D]$.

Proposition 4.5. *In $K\big(X^{[n+1,n]}\big)$, we have the identity: $\nu^*E^{[n+1]} = \lambda^*E^{[n]} + \rho^*E \otimes F$.*

Proof. Let $\overline{\partial}^{\,\mathrm{rel}}_{E,n\times 1}$, $\overline{\partial}^{\,\mathrm{rel}}_{E,n}$ and $\overline{\partial}^{\,\mathrm{rel}}_{E,n+1}$ be relative holomorphic structures on E compatible with $J^{\mathrm{rel}}_{n\times 1}$, J^{rel}_{n} and J^{rel}_{n+1}. For each $(\underline{x}, p) \in S^nX \times X$, we consider the exact sequence (9) on $\big(W_{\underline{x},p}, J^{\mathrm{rel}}_{n\times 1,\underline{x},p}\big)$ for the holomorphic vector bundle $\big(E_{|W_{\underline{x},p}}, \overline{\partial}^{\,\mathrm{rel}}_{E,n\times 1,\underline{x},p}\big)$. Putting these exact sequences in families over $S^nX \times X$, and considering the restriction to $X^{[n+1,n]}$, we get an exact sequence $0 \longrightarrow \rho^*E \otimes G \longrightarrow A \longrightarrow B \longrightarrow 0$, where G is a complex line bundle on $X^{[n+1,n]}$ and A and B are two vector bundles on $X^{[n+1,n]}$ such that $\forall (\underline{x}, p) \in S^nX \times X$:

$$(10) \qquad A_{|\,\xi,\xi',\underline{x},p} = \Big(E^{[n+1]}_{|\,\xi'}, \overline{\partial}^{\,\mathrm{rel}}_{E,n\times 1,\underline{x},p}, J^{\mathrm{rel}}_{n\times 1,\underline{x},p}\Big), \quad B_{|\,\xi,\xi',\underline{x},p} = \Big(E^{[n]}_{|\,\xi}, \overline{\partial}^{\,\mathrm{rel}}_{E,n\times 1,\underline{x},p}, J^{\mathrm{rel}}_{n\times 1,\underline{x},p}\Big).$$

Now, Φ is given by $\Phi(\xi, \xi', \underline{u}, \underline{v}) = \big(\phi_{\underline{u},\underline{v}*}\xi, S^n\phi_{\underline{u},\underline{v}}(\underline{u}), \psi_{\underline{u},\underline{v}*}\xi', S^{n+1}\psi_{\underline{u},\underline{v}*}(\underline{v})\big)$. Thus

$$\nu^*E^{[n+1]}_{|\,\xi,\xi',\underline{x},p} = \Big(E^{[n+1]}_{|\,\psi_{\underline{x},\underline{x}\,\cup\,p*}\xi'}, \overline{\partial}^{\,\mathrm{rel}}_{E,n+1,S^{n+1}\psi_{\underline{x},\underline{x}\,\cup\,p}(\underline{x}\,\cup\,p)}, J^{\mathrm{rel}}_{n+1,S^{n+1}\psi_{\underline{x},\underline{x}\,\cup\,p}(\underline{x}\,\cup\,p)}\Big),$$
$$\lambda^*E^{[n]}_{|\,\xi,\xi',\underline{x},x} = \Big(E^{[n]}_{|\,\phi_{\underline{x},\underline{x}\,\cup\,p*}\xi}, \overline{\partial}^{\,\mathrm{rel}}_{E,n,S^n\phi_{\underline{x},\underline{x}\,\cup\,p}(\underline{x})}, J^{\mathrm{rel}}_{n,S^n\phi_{\underline{x},\underline{x}\,\cup\,p}(\underline{x})}\Big).$$

As in Proposition 4.3, the classes A and B in $K\big(X^{[n+1,n]}\big)$ are independent of the structures used to define them. If $J^{\mathrm{rel}}_{n\times(n+1)} = \check{J}^{\mathrm{rel}}_{n+1}$ and $\forall (\underline{x}, p) \in S^nX \times X$, $\overline{\partial}^{\,\mathrm{rel}}_{E,n\times 1,\underline{x},p} = \overline{\partial}^{\,\mathrm{rel}}_{E,n\times 1,\underline{x}\,\cup\,p}$, we can take $\psi_{\underline{u},\underline{v}} = \mathrm{id}$ in a neighbourhood of $\underline{v}$. Thus $A = \nu^*E^{[n+1]}$. On the other way, if $J^{\mathrm{rel}}_{n\times(n+1)} = \check{J}^{\mathrm{rel}}_{n}$ and $\forall (\underline{x}, p) \in S^nX \times X$, $\overline{\partial}^{\,\mathrm{rel}}_{E,n\times 1,\underline{x},p} = \overline{\partial}^{\,\mathrm{rel}}_{E,n,\underline{x}}$ in a neighbourhood of $\underline{x}$, we can take $\phi_{\underline{u},\underline{v}} = \mathrm{id}$ in a neighbourhood of $\underline{u}$. Thus $B = \lambda^*E^{[n]}$. This proves that $\nu^*E^{[n+1]} - \lambda^*E^{[n]} = \rho^*E \otimes G$ in $K\big(X^{[n+1,n]}\big)$. If $\mathbb{T}$ is the trivial complex line bundle on X, $\nu^*\mathbb{T}^{[n+1]} \simeq \lambda^*\mathbb{T}^{[n]} \oplus \rho^*\mathbb{T}$ on $X^{[n+1,n]} \backslash D$. Thus G is trivial outside D. This yields $PD(c_1(G)) = \mu[D]$, where $\mu \in \mathbb{Q}$ and the computation of μ is local, as in Lemma 4.4. This gives $\mu = -1$. $\square$

The same proof can be used for higher incidence varieties. If $n,\,n'\in\mathbb{N}$ with $n'>n$, let $\overline{\partial}^{\,\mathrm{rel}}_{E,n\times(n'-n)}$ be a relative holomorphic structure on E compatible with a relative integrable structure $J^{\mathrm{rel}}_{n\times(n'-n)}$.

DEFINITION 4.6. $I_E^{[n',n]}$ is the complex vector bundle on $X^{[n',n]}$ whose fiber $I_E^{[n',n]}{}_{|\,\xi,\xi',\underline{x},\underline{y}}$ is

$$\ker\Big[\ \big(E^{[n']}_{|\xi'},\overline{\partial}^{\,\mathrm{rel}}_{E,\,n\times(n'-n),\,\underline{x},\underline{y}},\ J^{\mathrm{rel}}_{n\times(n'-n),\,\underline{x},\underline{y}}\big)\longrightarrow\big(E^{[n]}_{|\xi},\overline{\partial}^{\,\mathrm{rel}}_{E,\,n\times(n'-n),\,\underline{x},\underline{y}},\ J^{\mathrm{rel}}_{n\times(n'-n),\,\underline{x},\underline{y}}\big)\ \Big].$$

Thus $I_E^{[n',n]}=\ker\big(\ A\longrightarrow B\ \big)$, where A and B are vector bundles on $X^{[n',n]}$ defined by similar formulae as in (10). The class of $I_E^{[n',n]}$ in $K\big(X^{[n',n]}\big)$ is independent of $J^{\mathrm{rel}}_{n\times(n'-n)}$ and $\overline{\partial}^{\,\mathrm{rel}}_{E,n\times(n'-n)}$. If $\lambda\colon X^{[n',n]}\longrightarrow X^{[n]}$ and $\nu\colon X^{[n',n]}\longrightarrow X^{[n]}$ are the usual maps, then $I_E^{[n',n]}=\nu^*E^{[n']}-\lambda^*E^{[n]}$ in $K\big(X^{[n',n]}\big)$. If $n'=n+1$, $I_E^{[n',n]}=\rho^*E\otimes F$, but there is no such nice description for other values of n'.

4.2. Holomorphic curves in symplectic fourfolds. Until now, we have only considered integrable structures in small open sets of (X,J). In the sequel, it will be necessary to construct pseudo-holomorphic curves in X with respect to almost-complex structures close to J. To do so we use the following theorem of Donaldson [**Do**], which is a symplectic version of Kodaira's imbedding theorem:

THEOREM 4.7. [**Do**] *Let (V,ω) be a symplectic manifold of dimension $2n$ such that ω is an integral class. If h is a lift of ω in $H^2(V,\mathbb{Z})$, then, for $k\gg 0$, the classes $PD(kh)\in H_{2n-2}(V,\mathbb{Z})$ can be realized by homology classes of symplectic submanifolds. More precisely, if J is an almost-complex structure on V compatible with ω, we can write for k large enough $PD(kh)=\big[S_k\big]$ where S_k is a J_k-holomorphic submanifold in V and $||J_k-J||_{C^0}\leq C/\sqrt{k}$.*

We apply this theorem to our situation:

COROLLARY 4.8. *Let (X,ω) be a symplectic compact fourfold and J an adapted almost-complex structure on X. Then there exist almost-complex structures $\big(J_i\big)_{1\leq i\leq N}$ arbitrary close to J and J_i-holomorphic curves $\big(C_i\big)_{1\leq i\leq N}$ such that for all i, the classes $\big[C_i\big]$ span $H_2(X,\mathbb{Q})$ and J_i is integrable in a neighbourhood of C_i.*

PROOF. We pick $\alpha_1,\dots,\alpha_N$ in $H^2(X,\mathbb{R})$ such that the $\omega+\alpha_i$'s are symplectic forms in $H^2(X,\mathbb{Q})$ which generate $H^2(X,\mathbb{Q})$. There exist almost-complex structures $\big(\widetilde{J}_i\big)_{1\leq i\leq N}$ adapted to $(\omega+\alpha_i)_{1\leq i\leq N}$ arbitrary close to J if the α_i's are small enough. For $m\gg 0$, the symplectic forms $m(\omega+\alpha_i)$ are integral and Donaldson's theorem applies. We obtain J_i-holomorphic curves $(C_i)_{1\leq i\leq N}$ where J_i is arbitrary close to $\widetilde{J}_i$ and $\big[C_i\big]=k_i(\omega+\alpha_i)$. Let U_i be a small neighbourhood of the zero section of $N_{C_i/X}$. We identify U_i with a neighbourhood of C_i in X. Since $\dim C_i=2$, $J_{i|\,C_i}$ is integrable. Since C_i is J_i-holomorphic, $N_{C_i/X}$ is naturally a complex vector bundle over the complex curve C_i, so that we can put a holomorphic structure on it. This gives an integrable structure J'_i on U_i such that $J'_{i|\,C_i}=J_{i|\,C_i}$. We glue together J_i and J'_i in an annulus around C_i. The resulting almost-complex structure can be chosen arbitrary close to J_i if U_i is small enough. □

4.3. Computation of the boundary operator. Our aim now is to study the multiplicative structure of $H^*(X^{[n]},\mathbb{Q})$. In order to apply the results of Section 4.2, we will suppose that (X,ω) is a symplectic compact fourfold, and J an adapted almost-complex structure.

DEFINITION 4.9.

(i) If $\mathbb{T}$ is the trivial line bundle on X, the boundary operator ∂ is defined by

$$\begin{aligned} \partial : \bigoplus_n H^*\big(X^{[n]}, \mathbb{Q}\big) &\longrightarrow \bigoplus_n H^{*+2}\big(X^{[n]}, \mathbb{Q}\big) \\ \big(\alpha_n\big)_{n\geq 0} &\longmapsto \big(c_1(\mathbb{T}^{[n]}) \cup \alpha_n\big)_{n\geq 0}. \end{aligned}$$

(ii) If $\mathbb{H} = \oplus_n H^*\big(X^{[n]}, \mathbb{Q}\big)$ and $A \in \mathrm{End}(\mathbb{H})$, the derivative A' of A is defined as the commutator $A' = [\partial, A] = \partial \circ A - A \circ \partial$.

A fundamental step in the study of the cohomology rings of Hilbert schemes is the explicit computation of ∂. It has been achieved by Lehn in [**Le**]. We now generalize Lehn's result in our situation:

THEOREM 4.10. *Let (X, ω) be a compact symplectic fourfold. Then:*

(i) $$\big[q'_n(\alpha), q_m(\beta)\big] = -nm\left\{q_{n+m}(\alpha\beta) - \frac{|n|-1}{2}\,\delta_{n+m,0}\left(\int_X c_1(X)\alpha\beta\right)\mathrm{id}\right\}$$

(ii) $$q'_n(\alpha) = n\,\mathfrak{L}_n(\alpha) - \frac{n(|n|-1)}{2}\,q_n\big(c_1(X)\alpha\big),$$

where $\mathfrak{L}_n(\alpha)$ is introduced in Definition 3.10.

PROOF. We adapt the proof of [**Le**]. First, we prove that property (i) implies (ii). Indeed, by Proposition 3.11, property (i), and the Nakajima relations of Theorem 3.5, we can check that the commutator between the right hand side of (ii) and $q_m(\beta)$ is the right hand side of (i). The representation of the Heisenberg algebra $\mathcal{H}\big(H^*(X, \mathbb{C})\big)$ in $\mathbb{H} \otimes_{\mathbb{Q}} \mathbb{C}$ is irreducible by Proposition 3.9. By Schur's lemma, any operator which commutes to the Nakajima operators is a scalar. Thus we conclude that the difference $q'_n(\alpha) - n\,\mathfrak{L}_n(\alpha) + \frac{n(|n|-1)}{2}\,q_n\big(c_1(X)\alpha\big)$ must be a scalar which is obviously zero for degree reasons.

We turn to the proof of (i). A crucial point is that correspondences behave well under derivation:

LEMMA 4.11. *Let $u \in H_*\big(X^{[n',n]}, \mathbb{Q}\big)$ and $u_* : H^*\big(X^{[n]}, \mathbb{Q}\big) \longrightarrow H^*\big(X^{[n']}, \mathbb{Q}\big)$ be the associated correspondence map. Then $\big(u_*\big)' = \Big[u \cap c_1\big(I_{\mathbb{T}}^{[n',n]}\big)\Big]_*$ where $I_{\mathbb{T}}^{[n',n]}$ is defined in 4.6.*

PROOF. Let $\lambda : X^{[n',n]} \longrightarrow X^{[n]}$ and $\nu : X^{[n',n]} \longrightarrow X^{[n']}$ be the usual maps. We have the identity $\nu^*\mathbb{T}^{[n']} - \lambda^*\mathbb{T}^{[n]} = I_{\mathbb{T}}^{[n',n]}$ in $K\big(X^{[n',n]}\big)$. Thus

$$\begin{aligned} \big(u_*\big)'\tau &= c_1\big(\mathbb{T}^{[n']}\big) \cup u_*\tau - u_*\big(c_1\big(\mathbb{T}^{[n]}\big) \cup \tau\big) \\ &= PD^{-1}\Big[\big(\nu_*(u \cap \lambda^*\tau)\big) \cap c_1\big(\mathbb{T}^{[n']}\big) - \nu_*\big(u \cap \lambda^*\big(c_1\big(\mathbb{T}^{[n]}\big) \cup \tau\big)\big)\Big] \\ &= PD^{-1}\Big[\nu_*\Big(u \cap \big[\big(\nu^*c_1\big(\mathbb{T}^{[n']}\big) - \lambda^*c_1\big(\mathbb{T}^{[n]}\big)\big) \cup \lambda^*\tau\big]\Big)\Big] \\ &= PD^{-1}\,\nu_*\Big(\big[u \cap c_1\big(I_{\mathbb{T}}^{[n',n]}\big)\big] \cap \lambda^*\tau\Big). \end{aligned}$$

$\square$

Using Lemma 4.11, (i) can be seen as the commutation of two correspondences. Let us explain the main steps in the computation of $\big[q'_{-i}(\alpha), q_j(\beta)\big]$. The strategy is the same as in the proof of Theorem 3.5; we will keep the same notations. By Lemma 4.11, $\big[q'_{-i}(\alpha), q_j(\beta)\big]$ is induced by the class

$$p_{13*}\bigg[p_{124}^*\big[Q^{[n+j,n]}\big]\,.\,\Big(p_{235}^*\big[Q^{[n+j,n+j-i]}\big]\cap p_{23}^*\,c_1\big(I_{\mathbb{T}}^{[n+j,n+j-i]}\big)\Big)\bigg]$$
$$-\,p_{13*}\bigg[\Big(p_{124}^*\big[Q^{[n,n-i]}\big]\cap p_{12}^*\,c_1\big(I_{\mathbb{T}}^{[n,n-i]}\big)\Big)\,.\,p_{235}^*\big[Q^{[n+j-i,n-i]}\big]\bigg],$$

the two terms being computed in $X^{[n]\times[n+j]\times[n-i+j]\times[1]\times[1]}$ and in $X^{[n]\times[n-i]\times[n-i+j]\times[1]\times[1]}$ with suitable relative integrable structures as in the proof of Theorem 3.5. The difference with the computation of $\big[q_{-i}(\alpha), q_j(\beta)\big]$ lies in the classes $c_1\big(I_{\mathbb{T}}^{[n+j,n+j-i]}\big)$ and $c_1\big(I_{\mathbb{T}}^{[n,n-i]}\big)$. The natural morphism $\phi^*\,p_{23}^*\,I^{[n+j,n+j-i]}_{\mathbb{T}\,|\,D_{ij}}\longrightarrow I^{[n,n-i]}_{\mathbb{T}\,|\,C_{ij}}$ is an isomorphism. Thus $\phi^*c_1\big(p_{23}^*\,I_{\mathbb{T}}^{[n+j,n+j-i]}\big)_{|\,D_{ij}} = c_1\big(p_{12}^*\,I_{\mathbb{T}}^{[n,n-i]}\big)_{|\,C_{ij}}$. Since the degrees of all the classes have been decreased by two, S is no longer negligible in the computations (for the definition of S, see the proof of Theorem 3.5). This problem is solved in [**Le**] by considering the following cycles $\widetilde{C}_{ij}$ and $\widetilde{D}_{ij}$:

$$\widetilde{C}_{ij}=\Big\{(\xi,\xi',\xi'',s,t)\in A\cap B \text{ such that either } s\neq t \text{ or } s=t \text{ and } \xi,\xi'' \text{ are curvilinear}\Big\}$$
$$\widetilde{D}_{ij}=\Big\{(\xi,\xi',\xi'',s,t)\in A\cap B \text{ such that either } s\neq t \text{ or } s=t \text{ and } \xi' \text{ is curvilinear}\Big\}$$

The isomorphism ϕ can be extended to $\widetilde{\phi}:\widetilde{C}_{ij}\xrightarrow{\ \simeq\ }\widetilde{D}_{ij}$. If $\widetilde{S}=u\big(\partial\widetilde{C}_{ij}\big)=v\big(\partial\widetilde{D}_{ij}\big)$, then $\dim\widetilde{S}\leq 2(2n-i+j+2)-4$. Since we work with homology classes of degree $2(2n-i+j+2)-2$, $\widetilde{S}$ can be neglected.

– In the case $i\neq j$, Γ contains $Q^{[n,n-i+j]}$ if $i>j$ (resp. $Q^{[n-i+j,n]}$ if $i<j$), and all other components have lower dimensions. Recall that, by Lemma 3.2, $H_{2(2n-i+j+1)}\big(Q^{[n,n-i+j]},\mathbb{Z}\big)\simeq\mathbb{Z}$. This proves that $\big[q'_{-i}(\alpha), q_j(\beta)\big]=\mu_{ij}\,q_{-i+j}(\alpha\beta)$, where $\mu_{ij}\in\mathbb{Q}$. The computation of μ_{ij} is a local question, so Lehn's argument gives $\mu_{ij}=ij$.

– If $i=j$, $\big[q'_{-i}(\alpha), q_j(\beta)\big]$ is induced by a class in $H_{2(2n+1)}\big(\Delta_{X^{[n]}}\times\Delta_X,\mathbb{Q}\big)$ which turns out to be of the form $\big[\Delta_{X^{[n]}}\big]\otimes c_n$, where $c_n\in H_2(X,\mathbb{Q})$ (see [**Le**], Proposition 4.13); c_n is no longer a local datum. We must now compute c_n. In the projective case, c_n lies in the Neron-Severi group $NS(X)$ of X. Since the intersection pairing on $NS(X)$ is non-degenerate, c_n is uniquely determined by the numbers $c_n\,.\,[C]$ where C runs overs a basis of $NS(X)$. Remark that this argument already fails if X is integrable but not projective. Indeed, the intersection pairing on $NS(X)$ is no longer non-degenerate for non-algebraic surfaces. In our situation, we will compute $c_n\,.\,[C]$ where C is a pseudo-holomorphic curve for an almost-complex structure $\widetilde{J}$ close to J, integrable in a neighbourhood of C. The intersection pairing on $H^2(X,\mathbb{Q})$ being non-degenerate, by Corollary 4.8, c_n is uniquely determined by the numbers $c_n\,.\,[C]$.

If $\gamma\in H^*(X)$ is a class of even degree, we define the vertex operators $\big(S_m(\gamma)\big)_{m\geq 0}$ by the formula $\sum_{m\geq 0}S_m(\gamma)\,t^m=\exp\Big(\sum_{n>0}\frac{(-1)^{n-1}}{n}\,q_n(\gamma)\,t^n\Big)$. Since γ is of even degree, the operators $\big(q_i(\gamma)\big)_{i>0}$ commute in the usual sense, and the definition of $S_m(\gamma)$ is valid.

Lemma 4.12 (In the integrable case, [**Na**], [**Gr**]). *Let $\widetilde{J}$ be an almost-complex structure close to J, C a $\widetilde{J}$-holomorphic curve. Suppose that $\widetilde{J}$ is integrable in a neighbourhood of C. Then $\big[C^{[n]}\big]=S_n\big([C]\big)\,.\,1$, where $[C]$ and $\big[C^{[n]}\big]$ are the cohomology classes of C and $C^{[n]}$ in $H^2(X,\mathbb{Q})$ and $H^{2n}\big(X^{[n]},\mathbb{Q}\big)$ respectively.*

Proof. Let U be a small neighbourhood of C such that $\widetilde{J}$ is integrable in U. Then, for $n',\, n\in\mathbb{N}$, $n'>n$, we can suppose that $X^{[n']}$, $X^{[n]}$ and $X^{[n',n]}$ are the usual Hilbert schemes with the integrable structure $\widetilde{J}$ over $S^{n'}U$, S^nU and $S^nU\times S^{n'-n}U$ respectively. Since Lemma 4.12 holds in $H_c^{2n}\big(U^{[n]},\mathbb{Q}\big)$, we are done. □

To obtain the value of $c_n\,.\,[C]$, we compute in two different ways $I=\displaystyle\int_{X^{[n]}}\big[X_0^{[n]}\big]\,.\,\big[\partial C^{[n]}\big]$, where $\partial C^{[n]}=C^{[n]}\cap\partial X^{[n]}$. Since $C^{[n]}$ and $\partial X^{[n]}$ intersect generically transversally, by Lemma 4.12,

$$\big[\partial C^{[n]}\big]=\big[C^{[n]}\big]\,.\,\big[\partial X^{[n]}\big]=\Big(S^n\big([C]\big)\,.\,1\Big)\,.\,\Big(-2\,c_1\big(\mathbb{T}^{[n]}\big)\Big)=-2\,S'_n\big([C]\big).1.$$

We have the general formula $\displaystyle\int_{X^{[n]}}\big[X_0^{[n]}\big]\,.\,\alpha=\int_X q_{-n}(1)\,.\,\alpha$, thus $I=-2\displaystyle\int_X q_{-n}(1)\,S'_n\big([C]\big)\,.\,1$. Now $S'_n\big([C]\big)\,.\,1$ is a linear combination of terms of the form $q_{i_1}\big([C]\big)\dots q'_{i_k}\big([C]\big)\dots q_{i_N}\big([C]\big)\,.\,1$ where the indices i_l are positive and $i_1+\dots+i_N=n$. When we apply $q_{-n}(1)$, we obtain

– If $N=1$ and $i_1=N$, $q_{-n}(1)\,q'_n\big([C]\big)\,.\,1=\big[q_{-n}(1),q'_n\big([C]\big)\big]\,.\,1=-\displaystyle\int_X c_n.[C]$.

– If $N>1$

$$\begin{aligned}q_{-n}(1)\,q_{i_1}\big([C]\big)\dots q'_{i_k}\big([C]\big)\dots q_{i_N}\big([C]\big)\,.\,1&=q_{i_1}\big([C]\big)\dots\big[q_{-n}(1),q'_{i_k}\big([C]\big)\big]\,q_{i_{k+1}}\big([C]\big)\dots q_{i_N}\big([C]\big)\,.\,1\\&=-n\,i_k\,q_{i_1}\big([C]\big)\dots q_{i_k-n}\big([C]\big)\,q_{i_{k+1}}\big([C]\big)\dots q_{i_N}\big([C]\big)\,.\,1\end{aligned}$$

This is zero, unless $N=2$, and in this case we obtain a multiple of $[C]^2$. This yields the formula $I=\frac{1}{n}\,c_n\,.\,[C]+\binom{n}{2}\,[C]^2$ (see [**Le**] Lemma 3.18 for the explicit computations of the constants).

On the other hand, I can be computed geometrically. Let $\Delta\colon C\longrightarrow C_0^{[n]}$ be the natural isomorphism. Remark that $X_0^{[n]}$ and $C^{[n]}$ intersect generically transversally, the intersection being $\Delta(C)$. Then $I=\displaystyle\int_{X^{[n]}}\big[X_0^{[n]}\big]\,.\,\big[C^{[n]}\big]\,.\,\big[\partial X^{[n]}\big]=\int_{X^{[n]}}\big[\Delta(C)\big]\,.\,\big[\partial X^{[n]}\big]=\int_{C^{[n]}}\big[\Delta(C)\big]\,.\,\big[\partial C^{[n]}\big]$ since $\deg_{\Delta(C)}\Big(\mathcal{O}_{X^{[n]}}\big(\partial X^{[n]}\big)_{\,|\,U^{[n]}}\Big)_{|\,C}=\deg_{\Delta(C)}\Big(\mathcal{O}_{C^{[n]}}\big(\partial C^{[n]}\big)\Big)_{|\,C}$. Thus $I=-n(n-1)\deg K_C$. Recall that $\widetilde{J}$ is integrable in U. By the adjunction formula, $K_C=K_{U|C}\otimes N_{C/U}$ and we obtain $\deg K_C=-c_1(X)\,.\,[C]+[C]^2$. This gives $c_n=-\frac{1}{2}\,n^2(n-1)c_1(X)$. □

Corollary 4.13. *When α runs through a basis of $H^*(X,\mathbb{C})$, the operators δ and $q_1(\alpha)$ generate $\mathbb{H}$ from the vector* 1.

Proof. If $m\geq 0$, $\big[q'_1(\alpha),q_m(1)\big]=-m\,q_{m+1}(\alpha)$. Thus we can obtain all the positive Nakajima operators from ∂ and q_1. □

5. The multiplicative structure of $H^*(X^{[n]},\mathbb{Q})$

Let (X,ω) be a symplectic compact fourfold. From now on, we will suppose that $b_1(X)=0$. If E is a complex vector bundle on X, we can define canonical elements $E^{[n]}$ in $K\big(X^{[n]}\big)$ as explained in section 4.1. If $0\longrightarrow E\longrightarrow F\longrightarrow G\longrightarrow 0$ is an exact sequence of complex vector bundles on X, we have $E^{[n]}-F^{[n]}+G^{[n]}=0$ in $K\big(X^{[n]}\big)$. Thus we have a map $K(X)\longrightarrow\displaystyle\bigoplus_n K\big(X^{[n]}\big)$ which sends E to $\big(E^{[n]}\big)_{n\geq 0}$.

LEMMA 5.1. *There exists a map* $\begin{array}{rcl} H^*(X,\mathbb{Q}) & \longrightarrow & \bigoplus_n H^*\big(X^{[n]},\mathbb{Q}\big) \\ \alpha & \longmapsto & \bigoplus_n G(\alpha,n) \end{array}$ *such that for all* $n\in\mathbb{N}$ *the diagram*

$$\begin{tikzcd} K(X)\otimes_{\mathbb{Z}}\mathbb{Q} \arrow[r, ".^{[n]}"] \arrow[d, "\mathrm{ch}"'] & K\big(X^{[n]}\big)\otimes_{\mathbb{Z}}\mathbb{Q} \arrow[d, "\mathrm{ch}"] \\ H^*(X,\mathbb{Q}) \arrow[r, "G(\,.\,,n)"'] & H^*(X^{[n]},\mathbb{Q}) \end{tikzcd}$$

commutes.

PROOF. The surjectivity of $\mathrm{ch}: K(X)\otimes_{\mathbb{Z}}\mathbb{Q}\longrightarrow H^*(X,\mathbb{Q})$ holds when $b_1(X)=0$. Let x in $K(X)\otimes_{\mathbb{Z}}\mathbb{Q}$ be such that $\mathrm{ch}(x)=0$. We prove by induction that $\forall n\geq 1$, $\mathrm{ch}(x^{[n]})=0$. Let $\lambda\colon X^{[n+1,n]}\longrightarrow X^{[n]}$ and $\nu\colon X^{[n+1,n]}\longrightarrow X^{[n+1]}$ be the canonical maps. Then, by Proposition 4.5, $\nu^*x^{[n+1]}=\lambda^*x^{[n]}+\rho^*x\,.\,[F]$. If $\mathrm{ch}\big(x^{[n]}\big)=0$, then $\nu^*\,\mathrm{ch}\big(x^{[n+1]}\big)=0$. Since ν is locally topologically equivalent to a generically bijective algebraic map over $X^{[n+1]}$, $\nu_*\nu^*=\mathrm{id}$, and we get $\mathrm{ch}\big(x^{[n+1]}\big)=0$. □

DEFINITION 5.2. Let $\alpha\in H^*(X,\mathbb{Q})$ be a homogeneous cohomology class. Then

(i) $G_i(\alpha,n)$ denotes the $(|\alpha|+2i)$-th component of $G(\alpha,n)$.

(ii) $\mathfrak{G}_i(\alpha)$ denotes the operator $\mathbb{H}$ which acts on $H^*(X^{[n]},\mathbb{Q})$ by cup product with $G_i(\alpha,n)$.

PROPOSITION 5.3. $\forall\alpha,\beta\in H^*(X,\mathbb{Q})$, $\forall k\in\mathbb{N}$, $\big[\mathfrak{G}_k(\alpha),q_1(\beta)\big]=\frac{1}{k!}\,q_1^{(k)}(\alpha\beta)$.

PROOF. This is a consequence of Theorem 4.10 (i), Proposition 4.5 and Lemma 4.11 (see [**Le**], Theorem 4.2 for the detailed computations). □

We can now state some significant results on the cohomology rings of Hilbert schemes of symplectic fourfolds. These results are known in the integrable case and are formal consequences of the various relations between $q_n(\alpha)$, ∂, $\mathfrak{L}_n(\alpha)$ and $\mathfrak{G}_i(\alpha)$ (see e.g. Theorem 2.1 in [**Li-Qi-Wa-3**]), even if there is a lot of nontrivial combinatorics involved in the proofs. Thus the following results are formal consequences of Theorem 3.5, Proposition 3.11, Theorem 4.10 and Proposition 5.3. Recall that (X,ω) is a symplectic compact fourfold which satisfies $b_1(X)=0$.

THEOREM 5.4. *The classes* $G_i(\alpha,n)$, *where* $0\leq i<n$ *and* α *runs through a fixed basis of* $H^*(X,\mathbb{Q})$, *generate the ring* $H^*\big(X^{[n]},\mathbb{Q}\big)$.

This result was initially proved in [**Li-Qi-Wa-2**] using vertex algebras. For other proofs, see [**Li-Qi-Wa-1**] and [**Li-Qi-Wa-3**].

THEOREM 5.5. *Let* $\alpha_1,\dots,\alpha_s$ *be homogeneous classes in* $H^*(X,\mathbb{Q})$, $n,k_1,\dots,k_s$ *positive integers. Then* $\prod_{d=1}^s G_{k_d}(\alpha_d,n)$ *is a universal combination of terms of the form*

$$\mathrm{sign}(\alpha,I)\,.\,\underbrace{q_1(1)\circ\cdots\circ q_1(1)}_{m\ \text{times}}\circ q_{I_1}\big(\tau_{N_1}(\varepsilon_1\alpha_{I_1})\big)\circ\cdots\circ q_{I_j}\big(\tau_{N_j}(\varepsilon_j\alpha_{I_j})\big),$$

where I *runs through the partitions of* $\{1,\dots,n\}$, j *is the number of blocks in* I, $\alpha_{I_k}=\prod_{i\in I_k}\alpha_i$, $\varepsilon_k\in\{1,c_1(X),c_1(X)^2,c_2(X)\}$, $\prod_{k=1}^j\alpha_{I_k}=\mathrm{sign}(\alpha,I)\prod_{i=1}^s\alpha_i$ *(for other notations, see page 140) such that if* $N_k=\sum_{i\in I_k}i$,

(i) $|I_k|-\dfrac{|\varepsilon_k|}{2}\leq 2+\sum_{i\in I_k}k_i$ *for* $1\leq k\leq j$,

(ii) $\displaystyle\sum_{r=1}^{j}\left(\frac{|\varepsilon_r|}{2}+N_r\right)-j=2\sum_{i=1}^{s}k_i$,

(iii) $\displaystyle\sum_{r=1}^{j}N_r+m=n$.

For the proof, see [**Li-Qi-Wa-3**]. These theorems show that the ring $H^*\big(X^{[n]},\mathbb{Q}\big)$ can be built by universal formulae starting from $H^*(X,\mathbb{Q})$, $c_1(X)$ and $c_2(X)$.

Corollary 5.6. *Let X, Y be two symplectic compact fourfolds with $b_1(X)=b_1(Y)=0$ such that there exists an isomorphism of rings $\phi: H^*(X,\mathbb{Q}) \xrightarrow{\simeq} H^*(Y,\mathbb{Q})$ satisfying $\phi(c_i(X))=c_i(Y)$, $i=1,2$. Then, $\forall n\geq 0$, $H^*\big(X^{[n]},\mathbb{Q}\big)\simeq H^*\big(Y^{[n]},\mathbb{Q}\big)$.*

This is a consequence of Theorem 5.3 and 5.5.

There is a geometrical approach to the ring structure of $H^*\big(X^{[n]},\mathbb{Q}\big)$ through orbifold cohomology. If J is an adapted almost-complex structure on X, S^nX is an almost-complex Gorenstein orbifold. We can therefore consider the associated Chen-Ruan (or *orbifold*) cohomology ring $H^*_{CR}\big(S^nX,\mathbb{Q}\big)$ which is $\mathbb{Z}$-graded and depends only on the deformation class of J (see [**Ch-Ru**], [**Ad-Le-Ru**], [**Fa-Gö**]).

After works by Lehn-Sorger and Li-Qin-Wang, Qin and Wang developed a set of axioms which characterize $H^*_{CR}\big(S^nX,\mathbb{Q}\big)$ as a ring (see [**Ad-Le-Ru**]):

Theorem 5.7. [**Qi-Wa**] *Let A be a graded unitary ring and (X,J) an almost-complex compact fourfold. We suppose that*

(i) *A is an irreducible $\mathcal{H}\big(H^*(X,\mathbb{C})\big)$-module and 1 is a highest weight vector.*

(ii) *$\forall\alpha\in H^*(X,\mathbb{C})$, $\forall i\in\mathbb{N}$, there exist classes $O_i(\alpha,n)\in A^{|\alpha|+2i}$ such that if $\mathfrak{D}_i(\alpha)$ is the operator of product by $\bigoplus_n O_i(\alpha,n)$ and $\mathfrak{D}_1(1)=\partial$ is the derivation,*

(1) *$\forall\alpha,\beta\in H^*(X,\mathbb{C})$, $\forall k\in\mathbb{N}$, $\big[\mathfrak{D}_k(\alpha),q_1(\beta)\big]=q_1^{(k)}(\alpha\beta)$.*

(2) $\displaystyle\sum_{l_1+l_2+l_3=0} : q_{l_1}q_{l_2}q_{l_3} : (\tau_3 1_X)=-6\,\partial$ *(for the notations, see page 140).*

Then A is isomorphic as a ring to $H^*_{CR}\big(S^nX,\mathbb{C}\big)$.

We apply this theorem to prove Ruan's conjecture for the symmetric products of a symplectic fourfold.

Theorem 5.8. *Let (X,ω) be a symplectic compact fourfold such that $b_1(X)=0$ and $c_1(X)=0$. Then Ruan's crepant conjecture holds for S^nX, i.e. the rings $H^*\big(X^{[n]},\mathbb{Q}\big)$ and $H^*_{CR}\big(S^nX,\mathbb{Q}\big)$ are isomorphic.*

Proof. Let $O_k(\alpha,n)=k!\,\mathfrak{S}_k(\alpha,n)$. Then (1) is exactly Proposition 5.3. The relation (2) is a formal consequence of the Nakajima relations and of the formulae $\big[q'_n(\alpha),q_m(\beta)\big]=-nm\,q_{n+m}(\alpha\beta)$, $q'_n(\alpha)=n\,\mathfrak{L}_n(\alpha)$. $\square$

In the algebraic context, X is a $K3$ surface or an Enriques surface and the result of the theorem has been proved by Lehn and Sorger in [**Le-So-2**] (see also [**Qi-Wa**], Theorem 5.11).

6. The cobordism class of $X^{[n]}$

In this section, (X,J) is an almost-complex compact fourfold, no symplectic hypotheses being required. The almost-complex Hilbert schemes $X^{[n]}$ are endowed with a stable almost complex structure (see [**Vo 1**]), hence define almost-complex cobordism classes. By results of Novikov [**No**] and Milnor [**Mi**], the almost-complex cobordism class of $X^{[n]}$ is completely determined by the Chern numbers $\displaystyle\int_{X^{[n]}} P\big(c_1(X^{[n]}),\dots,c_{2n}(X^{[n]})\big)$ where P runs through all polynomials $P\in\mathbb{Q}\big[T_1,\dots,T_{2n}\big]$ of weighted degree $4n$, where $\deg(T_k)=2k$. Our aim is to prove the following result:

Theorem 6.1. *The almost-complex cobordism class of $X^{[n]}$ depends only on the almost-complex cobordism class of X.*

This means that if P is a weighted polynomial in $\mathbb{Q}\big[T_1,\dots,T_{2n}\big]$ of degree $4n$, there exists a weighted polynomial $\widetilde{P}\big[T_1,T_2\big]$ of degree 4, depending only on P and n, such that

$$\int_{X^{[n]}} P\big(c_1(X^{[n]}),\dots,c_{2n}(X^{[n]})\big)=\int_X \widetilde{P}\big(c_1(X),c_2(X)\big).$$

This result has been proved by Ellinsgrud, Göttsche and Lehn in [**El-Gö-Le**] when X is projective. We adapt their proof in our situation.

Let J_n^{rel} be a relative integrable structure in a neighbourhood W of Z_n satisfying the additional conditions listed in [**Vo 1**] page 711. Then $\big(X^{[n]},J_n^{\mathrm{rel}}\big)$ is smooth. If J_n^{rel} is arbitrary, by Proposition 2.3, $\big(X^{[n]},J_n^{\mathrm{rel}}\big)$ is only a topological manifold. Thus $TX^{[n]}$ does not exist any longer. However, the construction of the almost-complex structure on $X^{[n]}$ done in [**Vo 1**] shows that $\Big[TX^{[n]}\Big]=\Big[T^{\mathrm{rel}}W^{[n]}_{\mathrm{rel}|X^{[n]}}\Big]$ in $K\big(X^{[n]}\big)$. The advantage of using $T^{\mathrm{rel}}W^{[n]}_{\mathrm{rel}}$ is that this complex vector bundle is defined for *any* relative integrable complex structure J_n^{rel}. Let $\kappa_n=\big[T^{\mathrm{rel}}W^{[n]}_{\mathrm{rel}}\big]\in K\big(W^{[n]}_{\mathrm{rel}},J_n^{\mathrm{rel}}\big)$. Then $\kappa_{n|(X^{[n]},J_n^{\mathrm{rel}})}\in K\big(X^{[n]},J_n^{\mathrm{rel}}\big)\simeq K\big(X^{[n]}\big)$ and the image is independent of J_n^{rel}. Thus, for any relative integrable structure, $\kappa_{n|(X^{[n]},J_n^{\mathrm{rel}})}=\Big[TX^{[n]}\Big]$.

The main idea in [**El-Gö-Le**] is to use the diagram

$$\begin{tikzcd}
 & X^{[n+1,\,n]}\times X^m \arrow[dl, "(\nu,\mathrm{id})"'] \arrow[dr, "(\lambda,\rho,\mathrm{id})"] & \\
X^{[n+1]}\times X^m & & \big(X^{[n]}\times X\big)\times X^m
\end{tikzcd}$$

Then, for every cohomology class α on $X^{[n+1]}\times X^m$,

$$\int_{X^{[n+1]}\times X^m}\alpha=\frac{1}{n+1}\int_{X^{[n]}\times X^{m+1}}(\lambda,\rho,\mathrm{id})_*(\nu,\mathrm{id})^*\alpha.$$

Therefore, after practising n times this operation, it is possible to express $\int_{X^{[n]}}\alpha$ as $\int_{X^n}\widetilde{\alpha}$ where $\widetilde{\alpha}$ is obtained by successive pull-backs and push-forwards by (ν,id) and $(\lambda,\rho,\mathrm{id})$ for $m=1,\dots,n$. In order to apply this formula, it is essential to compare $\nu^*TX^{[n+1]}$ and $\lambda^*TX^{[n]}$ in $K\big(X^{[n+1,n]}\big)$. In the almost-complex situation, we can use relative tangent bundles $TW^{[n]}_{\mathrm{rel}}$ and $TW^{[n+1]}_{\mathrm{rel}}$, but we must perform homotopies between the various relative integrable structures involved. These homotopies do not change the K-theory classes.

Another difficulty appears. In the integrable case, $TX^{[n]} = pr_{1*}\mathcal{H}om_{\mathcal{O}_{X^{[n]}\times X}}\big(\mathcal{J}_n, \mathcal{O}_n\big)$, where $\mathcal{J}_n$ is the ideal sheaf of the incidence locus $Y_n \subseteq X^{[n]} \times X$ and $\mathcal{O}_n$ is the structure sheaf of Y_n. This shows the necessity of considering in this problem coherent sheaves and not only vector bundles.

In Appendix 7, we develop a general formalism for relatively coherent sheaves on spaces $\mathfrak{X}/B$ fibered in smooth analytic sets over a differentiable basis B with quotient singularities, such as $W_{\mathrm{rel}}^{[n]}$. These spaces carry a sheaf $\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}$ which is the sheaf of smooth functions holomorphic in the fibers. Intuitively, a relatively coherent sheaf $\mathcal{F}$ on $\mathfrak{X}/B$ is a family of coherent sheaves $\big(\mathcal{F}_b\big)_{b\in B}$ on $\big(\mathfrak{X}_b\big)_{b\in B}$ varying smoothly with b. If we take relative holomorphic coordinates on $\mathfrak{X}$, the local model for $\mathfrak{X}$ is $Z \times V$, where Z is a smooth analytic set and V is an open subset of the base B. Then the local model for a relatively coherent sheaf on $Z \times V$ is $pr_1^{-1}\mathcal{G} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{C}_V^{\infty}$, where $\mathcal{G}$ is a coherent analytic sheaf on Z. The usual operations on coherent sheaves, such that pull-back, push-forward and the associated derived operations, can be performed on relatively coherent sheaves for smooth morphisms holomorphic in the fibers satisfying some triviality conditions (in the case of the push-forward, there is also a triviality condition for the sheaf itself). In the sequel, it will be necessary to consider coherent sheaves in K-theory. This is achieved by the following proposition:

Proposition 6.2. *If $\mathcal{F}$ is a relatively coherent sheaf on $\mathfrak{X}/B$ and $\mathfrak{X}'$ is relatively compact in $\mathfrak{X}$, then $\mathcal{F}^{\infty} := \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}} \mathcal{C}_{\mathfrak{X}}^{\infty}$ admits a resolution on $\mathfrak{X}'$ by complex vector bundles $\big(E_i\big)_{1\le i\le N}$. The element $[\mathcal{F}^{\infty}] := \sum_{i=1}^{N}(-1)^{i-1}[E_i]$ in $K(\mathfrak{X}')$ is independent of $E_{\bullet}$ and depends only on the class of $\mathcal{F}$ in $K^{\mathrm{rel}}(\mathfrak{X})$, where $K^{\mathrm{rel}}(\mathfrak{X})$ is the Grothendieck group of the abelian category of relatively coherent sheaves with strict morphisms (see Definition 7.11).*

For the proof, see Corollary 7.8, Proposition 7.9 and Proposition 7.15 in Appendix 7.

The usual operations $f_!$ and $f^{\dagger}$ in analytic K-theory as well as the product $\mathcal{F}\,.\,\mathcal{G}$ and the dual morphism $\mathcal{F} \longrightarrow \mathcal{F}^{\vee}$ still exist in relative K-theory (see page 164).

We are now going to introduce the relatively coherent sheaf $\mathcal{O}_n^{\mathrm{rel}}$ which will be a fundamental object in the proof of Theorem 6.1. Next, we will compute its Chern classes in Proposition 6.3.

Let W be a small neighbourhood of $Z_{n,1}$ in $(S^nX \times X) \times X$ and $J_{n\times 1}^{\mathrm{rel}}$ a relative integrable complex structure on W such that $\forall \underline{x} \in S^nX$, $\forall p \in X$, $J_{n+1,\underline{x},p}^{\mathrm{rel}} = J_{n,\underline{x}}^{\mathrm{rel}}$ in a neighbourhood of $\underline{x}$, where J_n is a relative integrable structure in a neighbourhood of Z_n (the definitions of the incidence varieties are given by (5) and (6)). Let $\mathfrak{X} = \big(W_{\mathrm{rel}}^{[n+1,\,n]}, J_{n\times 1}^{\mathrm{rel}}\big)$ and $\mathfrak{X}' = \big(W_{\mathrm{rel}}^{[n]\times[1]}, J_{n\times 1}^{\mathrm{rel}}, J_{n\times 1}^{\mathrm{rel}}\big)$. These spaces lie above $S^nX \times X$. We have a natural map $\sigma^{\mathrm{rel}}\colon \mathfrak{X} \longrightarrow \mathfrak{X}'$ given by $\sigma^{\mathrm{rel}}(\xi, \xi', \underline{x}, p) = (\xi, \rho(\xi, \xi'), \underline{x}, p)$. Let $Y_n \subseteq \mathfrak{X}'$ be the incidence locus defined by $Y_n = \big\{(\xi, w, \underline{x}, p)$ such that $w \in \mathrm{supp}(\xi)\big\}$ and $D_{\mathrm{rel}} \subseteq \mathfrak{X}$ be the relative exceptional divisor defined by $D_{\mathrm{rel}} = \big\{(\xi, \xi', \underline{x}, p)$ such that $\rho_{\mathrm{rel}}(\xi, \xi') \in \mathrm{supp}(\xi)\big\}$. Taking relative holomorphic coordinates for $J_{n\times 1}^{\mathrm{rel}}$, we see that there exist two relatively coherent sheaves $\mathcal{O}_n^{\mathrm{rel}}$ and $\mathcal{J}_n^{\mathrm{rel}}$ on $\mathfrak{X}'$ and a relative holomorphic vector bundle $\mathcal{O}^{\mathrm{rel}}(-D_{\mathrm{rel}})$ on $\mathfrak{X}$ such that $\forall (\underline{x}, p) \in S^nX \times X$, $i_{(\underline{x},p)}^{*}\mathcal{O}_n^{\mathrm{rel}} = \mathcal{O}_{Y_{n,(\underline{x},p)}}$, $i_{(\underline{x},p)}^{*}\mathcal{J}_n^{\mathrm{rel}} = \mathcal{J}_{Y_{n,(\underline{x},p)}}$ and $i_{(\underline{x},p)}^{*}\mathcal{O}^{\mathrm{rel}}(-D_{\mathrm{rel}}) = \mathcal{O}_{W_{\underline{x},p}^{[n+1,\,n]}}\big(-(D_{\mathrm{rel}})_{\underline{x},p}\big)$. Remark that $\mathrm{supp}(\mathcal{O}_n) = Y_n$. Let

$$l = c_1\big(\mathcal{O}^{\mathrm{rel}}(-D_{\mathrm{rel}})\big)_{|X^{[n+1,n]}} \in H^2\big(X^{[n+1,n]}, \mathbb{Q}\big) \tag{11}$$

and let $\sigma\colon X^{[n+1,n]} \longrightarrow X^{[n]} \times X$ be the restriction of $\sigma^{\,\mathrm{rel}}$ to $X^{[n+1,n]}$.

PROPOSITION 6.3. ([**El-Gö-Le**] Lemma 1.1) *For all i in $\mathbb{N}^*$,* $\sigma_*(l^i) = (-1)^i\, c_i\big(\mathcal{O}_n^{\mathrm{rel},\infty}\big)_{|X^{[n]}\times X}$.

PROOF. In the integrable case, $\mathcal{J}_{Y_n}$ has homological depth 1. This proves that in our case $\mathcal{J}_n^{\mathrm{rel},\infty}$ locally admits a free resolution of length 2 on $\mathfrak{X}'$, hence, by Proposition 7.9, a global locally-free resolution of length 2. We denote it by $0 \longrightarrow A \longrightarrow B \longrightarrow \mathcal{J}_n^{\mathrm{rel},\infty} \longrightarrow 0$. Let $\pi : \mathbb{P}(B^*) \longrightarrow \mathfrak{X}'$ be the projective bundle of B^*, $\widetilde{\mathbb{P}}(B^*) = \mathbb{P}(B^*) \times_{S^nX\times X} (X^n \times X)$, δ the natural map $\widetilde{\mathbb{P}}(B^*) \longrightarrow \mathbb{P}(B^*)$ and s the section of $\pi^*A^* \otimes_{\mathcal{C}^\infty_{\mathbb{P}(B^*)}} \mathcal{C}^\infty_{\mathbb{P}(B^*)}(1)$ given by the morphism $\pi^*A \longrightarrow \pi^*B \longrightarrow \mathcal{C}^\infty_B(1)$.

LEMMA 6.4.

(i) *The zero locus of s is canonically isomorphic to $W_{\mathrm{rel}}^{[n+1,\,n]}$.*

(ii) *The section δ^*s of $\delta^*\big(\pi^*A^* \otimes_{\mathcal{C}^\infty_{\mathbb{P}(B^*)}} \mathcal{C}^\infty_{\mathbb{P}(B^*)}(1)\big)$ is transverse to the zero section.*

PROOF. (i) We take holomorphic relative coordinates above $V \subseteq S^nX \times X$ for W. This gives trivializations $\Omega \simeq \widetilde{W}^{[n+1,\,n]} \times V$ and $\Omega' \simeq \widetilde{W}^{[n]} \times \widetilde{W} \times V$ for open sets Ω and Ω' in $\mathfrak{X}$ and $\mathfrak{X}'$,where $\widetilde{W}$ is an open set in $\mathbb{C}^2$. We consider a free resolution $0 \longrightarrow \mathcal{O}^r_{\widetilde{W}^{[n]}\times\widetilde{W}} \longrightarrow \mathcal{O}^{r+1}_{\widetilde{W}^{[n]}\times\widetilde{W}}$ of the ideal sheaf $\mathcal{J}_{Y_n}$ on $\widetilde{W}^{[n]} \times \widetilde{W}$. We can suppose that the associated local resolution of $\mathcal{J}_n^{\mathrm{rel},\infty}$ on Ω' is a sub-resolution of the global one. If $\mathbb{T} = \mathcal{C}^\infty_{\widetilde{W}^{[n]}\times\widetilde{W}\times V}$, we have a diagram where all lines and columns are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{T}^r \arrow[r, "M"] \arrow[d] & \mathbb{T}^{r+1} \arrow[r] \arrow[d] & \mathcal{J}^{\mathrm{rel},\infty}_{n|\Omega'} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d] & B \arrow[r] \arrow[d] & \mathcal{J}^{\mathrm{rel},\infty}_{n|\Omega'} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{T}^m \arrow[r, "Q"] \arrow[d] & \mathbb{T}^m \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

Here, $M \in M_{r+1,r}\Big(\mathcal{O}_{\widetilde{W}^{[n+1,\,n]}}\Big)$ and $Q \in GL_m\Big(\mathcal{C}^\infty_{\widetilde{W}^{[n]}\times\widetilde{W}\times V}\Big)$. We can suppose that $Q = \mathrm{id}$. We take a global splitting $A \simeq \mathbb{T}^r \oplus \mathbb{T}^m$. This gives a splitting $B \simeq \mathbb{T}^{r+1} \oplus \mathbb{T}^m$ such that $\phi = \begin{pmatrix} M & 0 \\ 0 & \mathrm{id} \end{pmatrix}$. In local coordinates, the section s is given by $s(u,z,v) : (\alpha,\beta) \longmapsto u(M(z)\alpha,\beta)$ where $z \in \widetilde{W}^{[n]} \times \widetilde{W}$, $v \in V$, u lies in an affine chart of $\mathbb{P}^{r+m}(\mathbb{C})^*$, $\alpha \in \mathbb{C}^r$ and $\beta \in \mathbb{C}^m$. Then

$$Z(s) = \big\{(u,z,v) \text{ such that } u_{|\mathbb{C}^m} = 0 \text{ and } \forall \alpha \in \mathbb{C}^r,\ u\big(M(z)\alpha\big) = 0\big\}\cdot$$

In the integrable case, we know that $\widetilde{W}^{[n+1,\,n]} = \mathbb{P}\big(\mathcal{J}_{Y_n}\big)$ (see [**Da**]). Thus $Z(s) \simeq \widetilde{W}^{[n+1,\,n]} \times V$, which is isomorphic to Ω.

(ii) We can be more precise in the integrable case: $\mathbb{P}\big(\mathcal{J}_{Y_n}\big)$ is smooth and the section $\widetilde{s}$ used to define it is transverse to the zero section (see [**Ch**], [**Ti**]). Taking the same notations as above, if $u \in \mathbb{P}^{r+m}(\mathbb{C})^*$, $u_1 = u_{|\mathbb{C}^{r+1}}$ and $u_2 = u_{|\mathbb{C}^m}$, we have in local coordinates $s(u,z,v)(\alpha,\beta) = \big(\widetilde{s}(u_1,z,v)(\alpha), u_2(\beta)\big)$. This gives the result. $\square$

Now (ii) implies that the homology class of $Z(s)$ in $H_{8n+8}\big(\mathbb{P}(B^*),\mathbb{Q}\big)$ is Poincaré dual to the top Chern class of $\pi^*A^* \otimes_{\mathcal{C}^\infty_{\mathbb{P}(B^*)}} \mathcal{C}^\infty_{\mathbb{P}(B^*)}(1)$, since $\mathbb{P}(B^*)$ is rationally smooth. Let us consider the class $\varepsilon = c_1\big(\mathcal{C}^\infty_{\mathbb{P}(B^*)}(1)\big) \in H^2\big(\mathbb{P}(B^*),\mathbb{Q}\big)$. We have $PD^{-1}\Big(\big[W^{[n+1,\,n]}_{\mathrm{rel}}\big]\Big) = \displaystyle\sum_{k=0}^{d} \pi^* c_k(A^*)\,.\,\varepsilon^{d-k}$, where d is the complex rank of A. We consider now the diagram
$$\begin{tikzcd} \mathbb{P}(B^*) \arrow[d, "\pi"'] & \mathfrak{X} \arrow[l, hook', "j"'] \supseteq D_{\mathrm{rel}} \arrow[dl, "\sigma_{\mathrm{rel}}"] \\ \mathfrak{X}' & \end{tikzcd}$$
. The map σ_{rel} is proper. Hence
$$\begin{aligned} {\sigma_{\mathrm{rel}}}_*(\varepsilon^i) = PD^{-1}\big[\pi_*([\mathfrak{X}] \cap \varepsilon^i)\big] &= \sum_{k=0}^{d} c_k(A^*)\pi_*\big(\varepsilon^{d+i-k}\big) = \sum_{k=0}^{d} c_k(A^*) s_{i-k}(B^*) \\ &= c_i(A^* - B^*) = (-1)^i c_i\big(\mathcal{O}_n^{\mathrm{rel},\infty}\big) \qquad \text{(see [}\mathbf{Fu}\text{]).} \end{aligned}$$

To conclude, we use the diagram
$$\begin{tikzcd} \mathfrak{X} \arrow[d, "\sigma_{\mathrm{rel}}"'] & X^{[n+1,\,n]} \arrow[l, hook', "\iota"'] \arrow[d, "\sigma"] \\ \mathfrak{X}' & X^{[n]} \times X \arrow[l, hook'] \end{tikzcd}$$
and the formula $l = \iota^*\varepsilon$. $\square$

The classes $\mu_{i,n} := c_i\big(\mathcal{O}_n^{\mathrm{rel},\infty}\big)_{|X^{[n]}\times X} \in H^*\big(X^{[n]} \times X, \mathbb{Q}\big)$ will play an essential role in the sequel.

Let us now compute κ_n in terms of $\mathcal{O}_n^{\mathrm{rel}}$. We consider the map $p\colon W^{[n]}_{\mathrm{rel}} \times_{S^nX} W \longrightarrow W^{[n]}_{\mathrm{rel}}$ over S^nX. The important point is that p is not proper but that, if $Y_n \subseteq W^{[n]}_{\mathrm{rel}} \times_{S^nX} W$ is the relative incidence locus, $p_{|Y_n}$ is proper.

Proposition 6.5 ([**El-Gö-Le**] Proposition 2.2). *In $K^{\mathrm{rel}}\big(W^{[n]}_{\mathrm{rel}}, S^nX\big)$, we have*
$$\kappa_n = p_!\big(\mathcal{O}_n^{\mathrm{rel}} + \mathcal{O}_n^{\mathrm{rel}\,\vee} - \mathcal{O}_n^{\mathrm{rel}}\,.\,\mathcal{O}_n^{\mathrm{rel}\,\vee}\big).$$

Proof. We have $TW^{[n]}_{\mathrm{rel}} = p_*\,\mathrm{Hom}_{\mathcal{O}^{\mathrm{rel}}_{W^{[n]}_{\mathrm{rel}}\times_{S^nX} W}}\big(\mathcal{J}_n^{\mathrm{rel}}, \mathcal{O}_n^{\mathrm{rel}}\big)$ and the higher direct images vanish. Indeed, in the integrable case $TX^{[n]} = pr_{1*}\,\mathrm{Hom}_{\mathcal{O}_{X^{[n]}\times X}}\big(\mathcal{J}_n, \mathcal{O}_n\big)$ and pr_2 is flat on Y_n. Then we can apply the argument of [**El-Gö-Le**], Proposition 2.2, but we do not perform the last computation since p is not proper. $\square$

Let $\widetilde{W}$ be a neighbourhood of $Z_{n\times 1}$ and $J_{n\times 1}$ be a relative integrable complex structure on $\widetilde{W}$. Let us consider the following maps over $S^nX \times X$:

* $\rho_{\mathrm{rel}}\colon \widetilde{W}^{[n+1,\,n]}_{\mathrm{rel}} \longrightarrow \widetilde{W}_{\mathrm{rel}}$ is the residual map.

* $j = (\mathrm{id}, \rho_{\mathrm{rel}}) : \widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \longrightarrow \widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W}$.
* $\widetilde{p}$ is the first projection $\widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W} \longrightarrow \widetilde{W}^{[n+1,n]}_{\mathrm{rel}}$.
* If $f : \mathfrak{X} \longrightarrow \mathfrak{X}'$ (over $S^nX \times X$), we define $f_W = f \times_{S^nX\times X} \mathrm{id} : \mathfrak{X} \times_{S^nX\times X} \widetilde{W} \longrightarrow \mathfrak{X}' \times_{S^nX\times X} \widetilde{W}$.

$$\begin{tikzcd} \mathfrak{X} \arrow[rr, "f"] \arrow[dr] & & \mathfrak{X}' \arrow[dl] \\ & S^nX \times X & \end{tikzcd}$$

* $\widetilde{\psi} : \widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \longrightarrow \widetilde{W}^{[n+1]}_{\mathrm{rel}}$, $\quad \widetilde{\phi} : \widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \longrightarrow \widetilde{W}^{[n]}_{\mathrm{rel}}$.
* $\widetilde{\sigma} = (\widetilde{\phi}, \rho_{\mathrm{rel}}) : \widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \longrightarrow \widetilde{W}^{[n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W}$.

We introduce the following relatively coherent sheaves:

* $\widetilde{\mathcal{O}}^{\mathrm{rel}}_n$ and $\widetilde{\mathcal{O}}^{\mathrm{rel}}_{n+1}$ are the universal relative incidence structure sheaves on $\widetilde{W}^{[n]}_{\mathrm{rel}}$ and $\widetilde{W}^{[n+1]}_{\mathrm{rel}}$.
* $\mathcal{L} = \mathcal{O}^{\mathrm{rel}}(-D_{\mathrm{rel}})$, where $D_{\mathrm{rel}} \subseteq \widetilde{W}^{[n+1,n]}_{\mathrm{rel}}$ is the relative exceptional divisor.
* Δ_{rel} is the relative diagonal in $\widetilde{W}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W}_{\mathrm{rel}}$ and $\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}}$ is the associated structure sheaf.

Then we have the following properties which are immediate consequences of the same results in the integrable case:

(i) $j_*\mathcal{L} = \widetilde{p}^*\mathcal{L} \otimes \rho^*_{\mathrm{rel},W}\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}}$ on $\widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W}$.

(ii) We have an exact sequence on $\widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W}$:

$$0 \longrightarrow j_*\mathcal{L} \longrightarrow \widetilde{\psi}^*_W\, \widetilde{\mathcal{O}}^{\mathrm{rel}}_{n+1} \longrightarrow \widetilde{\phi}^*_W\, \widetilde{\mathcal{O}}^{\mathrm{rel}}_{n} \longrightarrow 0.$$

(iii) $\rho^*_{\mathrm{rel},W}\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}} = \rho^{\dagger}_{\mathrm{rel},W}\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}}$, $\quad \widetilde{\psi}^*_W\, \widetilde{\mathcal{O}}^{\mathrm{rel}}_{n+1} = \widetilde{\psi}^{\dagger}_W\, \widetilde{\mathcal{O}}^{\mathrm{rel}}_{n+1}$ and $\widetilde{\phi}^*_W\, \widetilde{\mathcal{O}}^{\mathrm{rel}}_{n} = \widetilde{\phi}^{\dagger}_W\, \widetilde{\mathcal{O}}^{\mathrm{rel}}_{n}$.

We will suppose that $J^{\mathrm{rel}}_{n\times 1}$ is of the form $J^{\mathrm{rel}}_{n\times 1, \underline{x}, p} = J^{\mathrm{rel}}_{n+1, \underline{x}\cup p}$. Then we have maps

$$\begin{tikzcd} \psi : \widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \arrow[r] \arrow[d] & \widetilde{W}^{[n+1]}_{\mathrm{rel}} \arrow[d] \\ S^nX \times X \arrow[r, "\cup"] & S^{n+1}X \end{tikzcd} \qquad \begin{tikzcd} \psi_W : \widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W} \arrow[r] \arrow[d] & \widetilde{W}^{[n+1]}_{\mathrm{rel}} \times_{S^{n+1}X} \widetilde{W} \arrow[d] \\ S^nX \times X \arrow[r, "\cup"] & S^{n+1}X \end{tikzcd}$$

and $\widetilde{\psi}^*_W\, \widetilde{\mathcal{O}}^{\mathrm{rel}}_{n+1} = \psi^*_W\, \mathcal{O}^{\mathrm{rel}}_{n+1}$. Let $\widetilde{\kappa}_n = \big[T\widetilde{W}^{[n]}_{\mathrm{rel}} \,/\, S^nX \times X\big] \in K^{\mathrm{rel}}\big(\widetilde{W}^{[n]}_{\mathrm{rel}}\big)$. We now prove the proposition:

Proposition 6.6. ([**El-Gö-Le**] Proposition 2.3) *In* $K^{\mathrm{rel}}\big(\widetilde{W}^{[n+1,n]}_{\mathrm{rel}}\big)$ *we have*

$$\begin{aligned}\psi^{\dagger}\kappa_{n+1} = \widetilde{\phi}^{\dagger}\widetilde{\kappa}_n + \mathcal{L} + \mathcal{L}^{\vee}.\, \rho^{\dagger}_{\mathrm{rel}}K^{\mathrm{rel}\,\vee}_W - \rho^{\dagger}_{\mathrm{rel}}\big(\mathcal{O}^{\mathrm{rel}}_W - T^{\mathrm{rel}}W + K^{\mathrm{rel}\,\vee}_W\big) \\ - \mathcal{L}\,.\,\widetilde{\sigma}^{\dagger}\widetilde{\mathcal{O}}^{\mathrm{rel}\,\vee}_n - \mathcal{L}^{\vee}.\, \rho^{\dagger}_{\mathrm{rel}}K^{\mathrm{rel}\,\vee}_W\,.\,\widetilde{\sigma}^{\dagger}\widetilde{\mathcal{O}}^{\mathrm{rel}}_n.\end{aligned}$$

Proof. By Proposition 6.5, $\psi^{\dagger}\kappa_{n+1} = \psi^{\dagger}p_!\big(\mathcal{O}^{\mathrm{rel}}_{n+1} + \mathcal{O}^{\mathrm{rel}\,\vee}_{n+1} - \mathcal{O}^{\mathrm{rel}}_{n+1}\,.\,\mathcal{O}^{\mathrm{rel}\,\vee}_{n+1}\big)$. We have the diagram

$$\begin{tikzcd} \widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W} \arrow[r, "\psi_W"] \arrow[d, "\widetilde{p}"'] & \widetilde{W}^{[n+1]}_{\mathrm{rel}} \times_{S^{n+1}X} \widetilde{W} \arrow[d, "p"] \\ \widetilde{W}^{[n+1,n]}_{\mathrm{rel}} \arrow[r, "\psi"] & W^{[n+1]}_{\mathrm{rel}} \end{tikzcd}$$

where the first column lies above $S^nX \times X$ and the second one above $S^{n+1}X$. We can apply the base change formulae as in the integrable case and we obtain:

$$
\begin{aligned}
\psi^{\dagger}\kappa_{n+1} &= \widetilde{p}_{\,!}\psi_W^{\dagger}\big(\mathcal{O}_{n+1}^{\mathrm{rel}} + \mathcal{O}_{n+1}^{\mathrm{rel}\vee} - \mathcal{O}_{n+1}^{\mathrm{rel}} \,.\, \mathcal{O}_{n+1}^{\mathrm{rel}\vee}\big) = \widetilde{p}_{\,!}\widetilde{\phi}_W^{\dagger}\big(\widetilde{\mathcal{O}}_n^{\mathrm{rel}} + \widetilde{\mathcal{O}}_n^{\mathrm{rel}\vee} - \widetilde{\mathcal{O}}_n^{\mathrm{rel}} \,.\, \widetilde{\mathcal{O}}_n^{\mathrm{rel}\vee}\big)\\
&\quad + \widetilde{p}_{\,!}\Big[\widetilde{p}^{\,\dagger}\mathcal{L} \,.\, \rho^{\dagger}_{\mathrm{rel},W}\,\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}} + \widetilde{p}^{\,\dagger}\mathcal{L}^{\vee} .\, \rho^{\dagger}_{\mathrm{rel},W}\,\mathcal{O}^{\mathrm{rel}\vee}_{\Delta_{\mathrm{rel}}} - \widetilde{p}^{\,\dagger}\big(\mathcal{L} \,.\, \mathcal{L}^{\vee}\big) \,.\, \rho^{\dagger}_{\mathrm{rel},W}\big(\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}} .\, \mathcal{O}^{\mathrm{rel}\vee}_{\Delta_{\mathrm{rel}}}\big)\\
&\qquad\qquad - \widetilde{p}^{\,\dagger}\mathcal{L} \,.\, \rho^{\dagger}_{\mathrm{rel},W}\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}} .\, \widetilde{\phi}_W^{\dagger}\,\widetilde{\mathcal{O}}_n^{\mathrm{rel}} - \widetilde{p}^{\,\dagger}\mathcal{L}^{\vee} .\, \rho^{\dagger}_{\mathrm{rel},W}\,\mathcal{O}^{\mathrm{rel}\vee}_{\Delta_{\mathrm{rel}}} .\, \widetilde{\phi}_W^{\dagger}\,\widetilde{\mathcal{O}}_n^{\mathrm{rel}}\Big]
\end{aligned}
$$

by properties (i) and (ii) page 155.

If q is the projection $\widetilde{W}^{[n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W} \longrightarrow \widetilde{W}^{[n]}_{\mathrm{rel}}$, then $\widetilde{\kappa}_n = q_{\,!}\big(\widetilde{\mathcal{O}}_n^{\mathrm{rel}} + \widetilde{\mathcal{O}}_n^{\mathrm{rel}\vee} - \widetilde{\mathcal{O}}_n^{\mathrm{rel}} \,.\, \widetilde{\mathcal{O}}_n^{\mathrm{rel}\vee}\big)$ so that by base change again, $\widetilde{\phi}^{\,\dagger}\widetilde{\kappa}_n = \widetilde{p}_{\,!}\widetilde{\phi}_W^{\dagger}\big(\widetilde{\mathcal{O}}_n^{\mathrm{rel}} + \widetilde{\mathcal{O}}_n^{\mathrm{rel}\vee} - \widetilde{\mathcal{O}}_n^{\mathrm{rel}} .\, \widetilde{\mathcal{O}}_n^{\mathrm{rel}\vee}\big)$. We also consider the diagram

$$
\begin{tikzcd}
\widetilde{W}^{[n+1,\,n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W} \arrow[r, "\rho_{\,\mathrm{rel},W}"] \arrow[d, "\widetilde{p}"'] & \widetilde{W} \times_{S^nX\times X} \widetilde{W} \arrow[d, "r"] \\
\widetilde{W}^{[n+1,\,n]}_{\mathrm{rel}} \arrow[r, "\rho_{\,\mathrm{rel}}"] & \widetilde{W}
\end{tikzcd}
$$

where all the terms are above $S^nX \times X$. This gives

$$
\begin{aligned}
\psi^{\dagger}\kappa_{n+1} &= \widetilde{\phi}^{\,\dagger}\widetilde{\kappa}_n + \mathcal{L} \,.\, \rho^{\dagger}_{\mathrm{rel}} r_{\,!}\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}} + \mathcal{L}^{\vee} .\, \rho^{\dagger}_{\mathrm{rel}} r_{\,!}\mathcal{O}^{\mathrm{rel}\vee}_{\Delta_{\mathrm{rel}}} - \rho^{\dagger}_{\mathrm{rel}} r_{\,!}\big(\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}} \,.\, \mathcal{O}^{\mathrm{rel}\vee}_{\Delta_{\mathrm{rel}}}\big)\\
&\quad - \mathcal{L} \,.\, \widetilde{p}_{\,!}\big(j_{\,!}\mathcal{O}^{\mathrm{rel}}_{W^{[n+1,\,n]}_{\mathrm{rel}}} .\, \widetilde{\phi}_W^{\dagger}\,\widetilde{\mathcal{O}}_n^{\mathrm{rel}\vee}\big) - \mathcal{L}^{\vee} .\, \widetilde{p}_{\,!}\Big[\big(\rho^{\dagger}_{\mathrm{rel},W}\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}}\big)^{\vee} .\, \widetilde{\phi}_W^{\dagger}\,\widetilde{\mathcal{O}}_n^{\mathrm{rel}}\Big].
\end{aligned}
$$

Now $r_{\,!}\,\mathcal{O}^{\mathrm{rel}}_{\Delta_{\mathrm{rel}}} = \mathcal{O}^{\mathrm{rel}}_{\widetilde{W}}$, $r_{\,!}\,\mathcal{O}^{\mathrm{rel}\vee}_{\Delta_{\mathrm{rel}}} = K^{\mathrm{rel}\vee}_{\widetilde{W}}$ and if $\delta\colon \widetilde{W} \longrightarrow \widetilde{W} \times_{S^nX\times X} \widetilde{W}$ is the diagonal injection, $\rho^{\dagger}_{\mathrm{rel},W}\,\mathcal{O}^{\mathrm{rel}\vee}_{\Delta_{\mathrm{rel}}} = \rho^{\dagger}_{\mathrm{rel},W}\,\delta_!\,K^{\mathrm{rel}\vee}_{\widetilde{W}} = j_{\,!}\rho^{\dagger}_{\mathrm{rel}}\,K^{\mathrm{rel}\vee}_{\widetilde{W}}$, thanks to the diagram

$$
\begin{tikzcd}
\widetilde{W}^{[n+1,\,n]}_{\mathrm{rel}} \arrow[r, "j"] \arrow[d, "\rho_{\,\mathrm{rel}}"] & \widetilde{W}^{[n+1,\,n]}_{\mathrm{rel}} \times_{S^nX\times X} \widetilde{W} \arrow[d, "\rho_{\,\mathrm{rel},W}"] \\
\widetilde{W} \arrow[r, "\delta"] & \widetilde{W} \times_{S^nX\times X} \widetilde{W}
\end{tikzcd}
$$

□

We want now to express in the cohomological setting all the identities we have obtained in K-theory. Recall that $\mu_{i,n} = c_i\big(\mathcal{O}_n^{\mathrm{rel},\infty}\big)_{|X^{[n]}\times X} \in H^{2i}\big(X^{[n]} \times X, \mathbb{Q}\big)$. By Lemma 7.16, $\mu_{i,n}$ is independent of the relative structure chosen on $\widetilde{W}^{[n]\times[1]}_{\mathrm{rel}}$. We consider the following maps:

$$
\begin{tikzcd}
 & X & \\
 & X^{[n+1,\,n]} \arrow[u, "\rho"'] \arrow[dl, "\lambda"'] \arrow[dr, "\nu"] \arrow[dd, "\sigma"] & \\
X^{[n]} & & X^{[n+1]} \\
 & X^{[n]} \times X &
\end{tikzcd}
$$

Lemma 6.7.

(i) $c_i\big(\widetilde{\phi}^{\dagger}\widetilde{\kappa}_n\big)=\lambda^* c_i\big(X^{[n]}\big)$.

(ii) $c_i\Big[\big(\widetilde{\phi}_W^{\dagger}\,\widetilde{\mathcal{O}}_n^{\mathrm{rel}}\big)^{\infty}\Big]=(\lambda,\mathrm{id})^*\mu_{i,n}$.

(iii) $c_i\Big[\big(\widetilde{\sigma}^{\dagger}\,\widetilde{\mathcal{O}}_n^{\mathrm{rel}}\big)^{\infty}\Big]_{|\,X^{[n+1,\,n]}}=\sigma^*\mu_{i,n}$.

(iv) $c_i\big(\rho_{\mathrm{rel}}^*\,T^{\mathrm{rel}}\widetilde{W}\big)_{|\,X^{[n+1,\,n]}}=\rho^* c_i(X)$.

(v) $c_i\Big[\big(\rho_{\mathrm{rel},W}^{\dagger}\,\mathcal{O}_{\Delta_{\mathrm{rel}}}^{\mathrm{rel}}\big)^{\infty}\Big]_{|\,X^{[n+1,\,n]}\times X}=(\rho,\mathrm{id})^* c_i\big(\mathcal{C}_{\Delta_X}^{\infty}\big)$.

Proof. The first three points are consequences of Lemma 7.16 in the appendix. For (iv), we consider the map $\delta\colon X\longrightarrow \widetilde{W}_{\mathrm{rel}}$ given by $\delta(x)=x\in \widetilde{W}_{\{x,\dots,x\},\,x}$. We have a diagram

$$\begin{tikzcd}
\widetilde{W}_{\mathrm{rel}}^{[n+1,\,n]} \arrow[r, "\rho_{\mathrm{rel}}"] & \widetilde{W} \\
X^{[n+1,\,n]} \arrow[u, hook] \arrow[r, "\rho"'] & X \arrow[u, hook, "\delta"']
\end{tikzcd}$$

When we restrict the family $\widetilde{W}/S^nX\times X$ to $\delta(X)\simeq X$, we obtain a neighbourhood $\widehat{W}$ of Δ_X in $X\times X$. Therefore $c_i\big(\rho_{\mathrm{rel}}^*\,T^{\mathrm{rel}}\widetilde{W}\big)_{|\,X^{[n+1,\,n]}}=\rho^*\delta^* c_i\big(T^{\mathrm{rel}}\,\widehat{W}\big)=\rho^* c_i(X)$.

For (v), we extend the structure $J_{n\times 1}^{\mathrm{rel}}$ to a relative integrable structure $J_{n\times 1\times 1}^{\mathrm{rel}}$ in a neighbourhood W of $Z_{n\times 1\times 1}$ such that for any $(\underline{x},p,q)\in S^nX\times X\times X$ near the incidence locus $p=q$, we have $J_{n\times 1\times 1,\underline{x},p,q}^{\mathrm{rel}}=J_{n\times 1,\underline{x},p}^{\mathrm{rel}}$. Then we consider the diagram

$$\begin{tikzcd}
\widetilde{W}^{[n+1,n]}\times_{S^nX\times X}\widetilde{W} \arrow[r, "\rho_{\mathrm{rel},W}"] \arrow[d, hook] & \widetilde{W}\times_{S^nX\times X}\widetilde{W} \arrow[d, hook] \\
W^{[n+1,n]\times 1} \arrow[r, "\rho_{\mathrm{rel}}\times\mathrm{id}"] & W\times_{S^nX\times X\times X} W \\
X^{[n+1,n]}\times X \arrow[u, hook] \arrow[r, "(\rho,\mathrm{id})"'] & X\times X \arrow[u, hook]
\end{tikzcd}$$

where the second line is taken over $S^nX\times X\times X$ with the relative integrable structure $J_{n\times 1\times 1}^{\mathrm{rel}}$ and the morphism $S^nX\times X\longrightarrow S^nX\times X\times X$ is the diagonal injection. This proves the result. ☐

Remark 6.8. By [**At-Hi**], $d_i:=c_i\big(\mathcal{C}_{\Delta_X}^{\infty}\big)\in H^{2i}(X\times X,\mathbb{Q})$ is a polynomial in $c_1(X)$ and $c_2(X)$.

Proposition 6.9.

(i) *$\forall i,n\in\mathbb{N}^*$, $(\nu,\mathrm{id})^*\mu_{i,n+1}-(\lambda,\mathrm{id})^*\mu_{i,n}=\displaystyle\sum_{k=0}^{i} pr_1^*\,l^k\,.\,(\rho,\mathrm{id})^*\,d_{i-k}$, where l is defined in* (11).

(ii) *$\forall i,n\in\mathbb{N}^*$, $\nu^* c_i\big(X^{[n+1]}\big)-\lambda^* c_i\big(X^{[n]}\big)$ is a universal polynomial in the classes l, $\rho^* c_i(X)$ and $\sigma^*\mu_{i,n}$.*

Proof. This is a consequence of Lemma 6.7, of relations (i) and (ii), page 155 and of Proposition 6.6 ☐

We are now going to perform the induction step.

Proposition 6.10. ([**El-Gö-Le**] Proposition 3.1) *If $n, m \in \mathbb{N}^*$, let P be a polynomial in the following classes on $X^{[n+1]} \times X^m$:*

$$pr_0^* c_i\big(X^{[n+1]}\big), \quad pr_{0k}^*\, \mu_{i,n+1}, \quad pr_{kl}^*\, d_i, \quad pr_k^* c_i(X), \qquad 1 \leq k, l \leq m.$$

Then there exists a polynomial $\widetilde{P}$ depending only on P, in the classes analogously defined on $X^{[n]} \times X^{m+1}$, such that $\displaystyle\int_{X^{[n+1]}\times X^m} P = \int_{X^{[n]}\times X^{m+1}} \widetilde{P}$.

Proof. We consider the incidence diagram

$$\begin{tikzcd} & X^{[n+1,\,n]} \times X^m \arrow[dl, "(\nu{,}\mathrm{id})"'] \arrow[dr, "(\sigma{,}\mathrm{id})"] & \\ X^{[n+1]} \times X^m & & \big(X^{[n]} \times X\big) \times X^m \end{tikzcd}$$

Since (ν, id) and (σ, id) are generically finite of degrees $n+1$ and 1,

$$\int_{X^{[n+1]}\times X^m} P = \frac{1}{n+1} \int_{X^{[n]}\times X^{m+1}} (\sigma, \mathrm{id})_* (\nu, \mathrm{id})^* P.$$

By Proposition 6.9 (ii), $(\nu, \mathrm{id})^*\, pr_0^*\, c_i\big(X^{[n+1]}\big) - (\sigma, \mathrm{id})^*\, pr_0^*\, c_i\big(X^{[n]}\big)$ is a polynomial in the classes $pr_0^*\, l$, $(\sigma, \mathrm{id})^*\, pr_1^*\, c_i(X)$ and $(\sigma, \mathrm{id})^*\, pr_{01}^*\, \mu_{i,n}$. By Proposition 6.3, $(\sigma, \mathrm{id})_*\, l^{\,i} = (-1)^i\, pr_{01}^*\, \mu_{i,n}$. Thus, $(\sigma, \mathrm{id})_*\, (\nu, \mathrm{id})^*\, pr_0^*\, c_i\big(X^{[n+1]}\big)$ is a polynomial in $pr_{01}^*\, \mu_{i,n}$ and $pr_1^*\, c_i(X)$. By Proposition 6.9 (i), $(\nu, \mathrm{id})^*\, pr_{0k}^*\, \mu_{i,n+1} - (\sigma, \mathrm{id})^*\, pr_{0,k+1}^*\, \mu_{i,n}$ is a polynomial in the classes $pr_0^*\, l$ and $(\sigma\, \mathrm{id})^*\, pr_{1k}^*\, d_i$. Then we can use Proposition 6.3 again. To conclude, we use the relations $(\nu, \mathrm{id})^*\, pr_{kl}\, d_i = (\sigma, \mathrm{id})^* pr_{k+1,\,l+1}\, d_i$ and $(\nu, \mathrm{id})^*\, pr_k^*\, c_i(X) = (\sigma, \mathrm{id})^*\, pr_{k+1}^*\, c_i(X)$. □

We can now finish the proof of Theorem 6.1. We write

$$\int_{X^{[n]}} P\Big(c_1\big(X^{[n]}\big), \ldots, c_{2n}\big(X^{[n]}\big)\Big) = \int_{X^{[n-1]}\times X} \widetilde{P}_1 = \int_{X^{[n-2]}\times X^2} \widetilde{P}_2 = \cdots = \int_{X^n} \widetilde{P}$$

where $\widetilde{P}$ is a polynomial in the classes $pr_k^*\, c_i(X)$ and $pr_{kl}^*\, d_i$. Since d_i is a polynomial in $c_1(X)$ and $c_2(X)$, we are done.

□

7. Appendix I: relative coherent sheaves

In this appendix, we develop a general formalism for relative coherent sheaves over a differentiable basis.

Let $\big(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}, \pi, B\big)$ be such that

- $\big(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}\big)$ is a ringed topological space,
- $B = M/_{G}$, where M is a smooth manifold, G is a finite subgroup of $\mathrm{Diff}(M)$, and B is endowed with the sheaf $\mathcal{C}_B^\infty = p_*\big(\mathcal{C}_M^\infty\big)^G$, where $p\colon M \longrightarrow B$ is the projection,
- $\pi\colon \mathfrak{X} \longrightarrow B$ is a continuous surjective map,

- $\forall x \in \mathfrak{X}$, there exist a neighbourhood U_x of x in $\mathfrak{X}$, a smooth analytic set Z and an isomorphism of ringed spaces

$$\begin{tikzcd} \big(U_x, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}\,|\,U_x}\big) \arrow[dr, "\pi"'] & & \big(Z\times V, \mathcal{O}^{\mathrm{rel}}_{Z\times V}\big) \arrow[ll, "\simeq"'] \arrow[dl, "pr_2"] \\ & \big(V, \mathcal{C}^{\infty}_V\big) & \end{tikzcd}$$

  where V is an open set of B and $\mathcal{O}^{\mathrm{rel}}_{Z\times V}$ is the sheaf of functions in $\mathcal{C}^{\infty}_{Z\times V}$ which are holomorphic in the first variable.

The sheaf $\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}$ is the sheaf of smooth functions on $\mathfrak{X}$ whis are holomorphic on the fibers of π. A typical example of such an objet is $(\mathfrak{X}, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}, \pi, B) = \big(W^{[n]}_{\mathrm{rel}}, \mathcal{O}^{\mathrm{rel}}_{W^{[n]}_{\mathrm{rel}}}, \pi, S^n X\big)$.

We consider a triplet $(\mathfrak{X}, \pi, B)$. For all $b \in B$, the fiber $\mathfrak{X}_b = \pi^{-1}(b)$ is a smooth analytic space. Our aim is to glue together coherent analytic sheaves on these fibers.

Definition 7.1. A sheaf $\mathcal{F}$ of $\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}$-modules is *relatively coherent* if, for all x in $\mathfrak{X}$ and for all trivialization $\phi\colon Z\times V \longrightarrow U_x$ of $\mathfrak{X}$ in a sufficiently small neighbourhood U_x of x, there exists a coherent analytic sheaf $\mathcal{G}$ on Z such that $\phi^{-1}\mathcal{F} \simeq pr_1^{-1}\mathcal{G} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}$ as sheaves of $\mathcal{O}^{\mathrm{rel}}_{Z\times V}$-modules.

Lemma 7.2. *If $\mathcal{F}$ and $\mathcal{G}$ are relatively coherent sheaves on $\mathfrak{X}$, so are $\mathcal{F}\otimes_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}\mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})$ as well as $\mathrm{Tor}^i_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})$ and $\mathcal{E}xt^i_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})$ for $i \geq 1$.*

Proof. Let $x \in \mathfrak{X}$ and let $\phi\colon U_x \longrightarrow Z\times V$ be a trivialization of $\mathfrak{X}$ on a neighbourhood U_x of x. Then $\phi^{-1}\mathcal{F} \simeq pr_1^{-1}\overline{\mathcal{F}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}$ and $\phi^{-1}\mathcal{G} \simeq pr_1^{-1}\overline{\mathcal{G}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}$. Therefore, since $\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}} \simeq \phi^{-1}\mathcal{O}^{\mathrm{rel}}_{Z\times V}$, we have

$$\phi^{-1}\big(\mathcal{F}\otimes_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}\mathcal{G}\big) \simeq pr_1^{-1}\big(\overline{\mathcal{F}}\otimes\overline{\mathcal{G}}\big) \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V},$$
$$\phi^{-1}\mathcal{H}om_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G}) \simeq \mathcal{H}om_{\mathcal{O}_Z}\big(\overline{\mathcal{F}},\overline{\mathcal{G}}\big) \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}$$

By Lemma 7.7 (ii), $\mathcal{O}^{\mathrm{rel}}_{Z\times V}$ is flat over $pr_1^{-1}\mathcal{O}_Z$ and we get

$$\phi^{-1}\big(\mathrm{Tor}^i_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G})\big) \simeq pr_1^{-1}\big(\mathrm{Tor}^i_{\mathcal{O}_Z}(\overline{\mathcal{F}},\overline{\mathcal{G}})\big) \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V},$$
$$\phi^{-1}\mathcal{E}xt^i_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}(\mathcal{F},\mathcal{G}) \simeq \mathcal{E}xt^i_{\mathcal{O}_Z}\big(\overline{\mathcal{F}},\overline{\mathcal{G}}\big) \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}$$

□

Lemma 7.3. *Let $f\colon \mathfrak{X}/B \longrightarrow \mathfrak{X}'/B'$ be a smooth map holomorphic in the fibers such that for all x in $\mathfrak{X}$ we can find trivializations around x and $f(x)$ in which $f : Z\times V \longrightarrow Z'\times V'$ is of the form $(z,v) \longmapsto (g(z), h(v))$. Then for any relatively coherent sheaf $\mathcal{F}$ on $\mathfrak{X}'$, the sheaf $f^*\mathcal{F} := f^{-1}\mathcal{F}\otimes_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}$ is relatively coherent, as well as $\mathrm{Tor}^i_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}}(f^{-1}\mathcal{F}, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}})$ for $i\geq 1$.*

Proof. We have

$$f^{-1}\Big(pr_1^{-1}\mathcal{G}\otimes_{pr_1^{-1}\mathcal{O}_{Z'}}\mathcal{O}^{\mathrm{rel}}_{Z'\times V'}\Big) \otimes_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{Z'\times V'}} \mathcal{O}^{\mathrm{rel}}_{Z\times V} = pr_1^{-1}(g^*\mathcal{G}) \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}.$$

By Lemma 7.7 (ii) again, we obtain

$$\mathrm{Tor}^i_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{Z'\times V'}}\Big(f^{-1}\big[pr_1^{-1}\mathcal{G}\otimes_{pr_1^{-1}\mathcal{O}_{Z'}}\mathcal{O}^{\mathrm{rel}}_{Z'\times V'}\big],\mathcal{O}^{\mathrm{rel}}_{Z\times V}\Big)\simeq pr_1^{-1}\,\mathrm{Tor}^i_{g^{-1}\mathcal{O}_{Z'}}\big(g^{-1}\mathcal{G},\mathcal{O}_Z\big)\otimes_{pr_1^{-1}\mathcal{O}_Z}\mathcal{O}^{\mathrm{rel}}_{Z\times V}.$$

□

Corollary 7.4. *If $b\in B$ and $i_b\colon\mathfrak{X}_b/\{b\}\longrightarrow\mathfrak{X}/B$ is the injection of the fiber $\mathfrak{X}_b$, then for any relatively coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$, $i_b^*\mathcal{F}$ is a coherent analytic sheaf on $\mathfrak{X}_b$.*

Proof. We apply Lemma 7.3 to i_b. Then $i_b^*\mathcal{F}$ is a relatively coherent sheaf on $\mathfrak{X}_b$. Since the basis of $\mathfrak{X}_b$ is reduced to the point $\{b\}$, $i_b^*\mathcal{F}$ is coherent on $\mathfrak{X}_b$. □

For direct images, it is necessary to consider trivializations in bigger open sets.

Definition 7.5. Let $f:\mathfrak{X}/B\longrightarrow\mathfrak{X}'/B$ be a smooth map over B holomorphic in the fibers and $\mathcal{F}$ be a relatively coherent sheaf on $\mathfrak{X}$. We say that $\mathcal{F}$ is *relatively coherent for f* if for all x' in $\mathfrak{X}'$ there exist a trivialization of $\mathfrak{X}'$ in a neighbourhood $U_{x'}$ of x' and a trivialization ϕ of $\mathfrak{X}$ in a neighbourhood of $f^{-1}(U_{x'})\cap\mathrm{supp}(\mathcal{F})$ in which $f:Z\times V\longrightarrow Z'\times V'$ is of the form $(z,v)\longmapsto(g(z),v)$ and $\phi^{-1}\mathcal{F}\simeq pr_1^{-1}\mathcal{G}\otimes_{pr_1^{-1}\mathcal{O}_Z}\mathcal{O}^{\mathrm{rel}}_{Z\times V}$ where $\mathcal{G}$ is coherent on Z and $g_{|\,\mathrm{supp}\,\mathcal{G}}$ is proper.

Lemma 7.6. *Let $f\colon\mathfrak{X}/B\longrightarrow\mathfrak{X}'/B$ be a smooth map over B holomorphic in the fibers and $\mathcal{F}$ be a relatively coherent sheaf for f on $\mathfrak{X}$. Then $f_*\mathcal{F}$ as well as the higher direct images $R^if_*\mathcal{F}$ are relatively coherent sheaves.*

Proof. We have a diagram

$$\begin{tikzcd} W \arrow[d, "f"'] & Z\times V \arrow[l, "\sim"', "\phi"] \arrow[d, "(g{,}\mathrm{id})"] \\ U_{x'} & Z'\times V \arrow[l, "\phi'"] \end{tikzcd}$$

where W is a neighbourhood of $f^{-1}(U_{x'})\cap\mathrm{supp}(\mathcal{F})$. Then

$$\begin{aligned}\phi'^{-1}R^if_*\,\mathcal{F}_{|f^{-1}(U_{x'})}&\simeq\phi'^{-1}R^if_*\,\mathcal{F}_{|W}\simeq R^i(g,\mathrm{id})_*\phi^{-1}\mathcal{F}\simeq R^i(g,\mathrm{id})_*\big[pr_1^{-1}\mathcal{G}\otimes_{pr_1^{-1}\mathcal{O}_Z}\mathcal{O}^{\mathrm{rel}}_{Z\times V}\big]\\&=pr_1^{-1}\big(R^ig_*\,\mathcal{G}\big)\otimes_{pr_1^{-1}\mathcal{O}_{Z'}}\mathcal{O}^{\mathrm{rel}}_{Z'\times V}.\end{aligned}$$

Since g is proper on $\mathrm{supp}(\mathcal{G})$, the sheaves $R^ig_*\,\mathcal{G}$ are coherent and we are done. □

We now focus our attention on the K-theoretical aspects of relatively coherent sheaves. If $\mathcal{F}$ is a relatively coherent sheaf on $\mathfrak{X}$, we define $\mathcal{F}^\infty:=\mathcal{F}\otimes_{\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}}\mathcal{C}^\infty_{\mathfrak{X}}$. Our aim is to construct locally free resolutions of $\mathcal{F}^\infty$ on any relatively compact open set in $\mathfrak{X}$. We need the following flatness lemma:

Lemma 7.7. *Let $U\subseteq\mathbb{R}^n$, G a finite group acting smoothly on U and Z a smooth analytic set. Then*

(i) *$\mathcal{C}^\infty_{Z\times U/G}$ is flat over $pr_1^{-1}\mathcal{O}_Z$.*

(ii) *$\mathcal{O}^{\mathrm{rel}}_{Z\times U/G}$ is flat over $pr_1^{-1}\mathcal{O}_Z$.*

PROOF. (i) Let $\delta : U \longrightarrow U/_G$ be the projection map and $\mathcal{M}$ a sheaf of $pr_1^{-1}\mathcal{O}_Z$-modules. Then

$$(\delta, \mathrm{id})^{-1}\Big(\mathcal{M} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{C}^\infty_{Z\times U/G}\Big) = (\delta, \mathrm{id})^{-1}\mathcal{M} \otimes_{pr_1^{-1}\mathcal{O}_Z} \big(\mathcal{C}^\infty_{Z\times U}\big)^G \simeq \Big[(\delta, \mathrm{id})^{-1} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{C}^\infty_{Z\times U}\Big]^G.$$

Since the functor $\mathcal{F} \longrightarrow \mathcal{F}^G$ is exact, it suffices to prove that $\mathcal{C}^\infty_{Z\times U}$ is flat over $pr_1^{-1}\mathcal{O}_Z$. We use the notation $\mathcal{C}^\omega$ for real analytic functions. Then $\mathcal{C}^\omega_Z$ is flat over $\mathcal{O}_Z$, $\mathcal{C}^\omega_{Z\times U}$ is flat over $pr_1^{-1}\mathcal{C}^\omega_Z$ and $\mathcal{C}^\infty_{Z\times U}$ is flat over $\mathcal{C}^\omega_{Z\times U}$ by a theorem of Malgrange ([**Ma**], see [**At-Hi**]).

(ii) As in (i), it suffices to prove that $\mathcal{O}^{\mathrm{rel}}_{Z\times U}$ is flat over $pr_1^{-1}\mathcal{O}_Z$. This follows from [**Ma**, Th. 2.bis]. $\square$

COROLLARY 7.8. *If $\mathcal{F}$ is a relatively coherent sheaf on $\mathfrak{X}$, then for all x in $\mathfrak{X}$, $\mathcal{F}^\infty$ admits a finite free resolution in a neighbourhood of x.*

PROOF. Let us write locally, $\phi^{-1}\mathcal{F} \simeq pr_1^{-1}\mathcal{G} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V}$. Then $\mathcal{G}$ admits locally a finite $\mathcal{O}_Z$-free resolution and we are done by Lemma 7.7. $\square$

We now prove a general result about $\mathcal{C}^\infty$ resolutions:

PROPOSITION 7.9. *Let M be a smooth manifold, G a finite group acting smoothly on M and $Y = M/_G$. Let $\mathcal{H}$ be a sheaf of $\mathcal{C}^\infty_Y$-modules which admits a finite free resolution in a neighbourhood of any point $y \in Y$. Then*

(i) *$\mathcal{H}$ admits a finite locally free resolution in a neighbourhood of any compact set of Y.*
(ii) *Two resolutions of $\mathcal{H}$ in a neighbourhood of a compact set are sub-resolutions of a third one.*

PROOF. We will use several times the following lemma:

LEMMA 7.10. *Let $Y = M/G$ and let $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ be an exact sequence of $\mathcal{C}^\infty_Y$-modules on an open set $U \subseteq Y$ such that $\mathcal{H}$ is locally free. Then this exact sequence globally splits.*

PROOF. It is sufficient to prove that $\mathrm{Ext}^1_{\mathcal{C}^\infty_Y}(U, \mathcal{H}, \mathcal{F}) = 0$. The $\mathcal{E}xt \Longrightarrow \mathrm{Ext}$ spectral sequence satisfies

$$\begin{cases} E_2^{p,q} = H^p\Big[U, \mathcal{E}xt^q_{\mathcal{C}^\infty_Y}(\mathcal{H}, \mathcal{F})\Big] \\ E_\infty^{p,q} = \mathrm{Gr}^p\, \mathrm{Ext}^{p+q}_{\mathcal{C}^\infty_Y}(U, \mathcal{H}, \mathcal{F}). \end{cases}$$

Since $\mathcal{H}$ is locally free, $\mathcal{E}xt^q(\mathcal{H}, \mathcal{F}) = 0$ for $q > 0$ and since $\mathcal{H}om_{\mathcal{C}^\infty_Y}(\mathcal{H}, \mathcal{F})$ is a fine sheaf, $E_2^{p,0} = 0$ for $p \geq 1$. Therefore all the terms $E_2^{p,q}$ vanish except $E_2^{0,0}$. This implies $Ext^1_{\mathcal{C}^\infty_Y}(U, \mathcal{H}, \mathcal{F}) = 0$. $\square$

Let us prove (i). Let $K \subseteq Y$ be a compact. We argue by induction on the least non zero integer N such that for all y in K, $\mathcal{H}$ admits a locally free resolution of length $\leq N$ in a neighbourhood of y. We choose a finite covering $\big(U_i\big)_{1\leq i\leq d}$ of K and open sets $\big(V_i\big)_{1\leq i\leq d}$ such that $\overline{U}_i \subseteq V_i$ and $\mathcal{H}_{|V_i}$ admits a finite free resolution of length $\leq N$:

$$0 \longrightarrow \big(\mathcal{C}^\infty_{V_i}\big)^{n_{iN}} \xrightarrow{M_{iN}} \cdots \xrightarrow{M_{i1}} \big(\mathcal{C}^\infty_{V_i}\big)^{n_{i1}} \xrightarrow{M_{i0}} \mathcal{H} \longrightarrow 0.$$

Let χ_i be a smooth function on Y such that $\chi_i = 1$ on U_i and $\mathrm{supp}(\chi_i) \subseteq V_i$. Then we have a complex of sheaves on Y, which is exact on U_i:

$$0 \longrightarrow \big(\mathcal{C}^\infty_Y\big)^{n_{iN}} \xrightarrow{\chi_i M_{iN}} \cdots \xrightarrow{\chi_i M_{i1}} \big(\mathcal{C}^\infty_Y\big)^{n_{i1}} \xrightarrow{\chi_i M_{i0}} \mathcal{H} \longrightarrow 0.$$

Let $\pi_i = \chi_i M_{i0}$. We define $E = \bigoplus_{i=1}^{d} \big(\mathcal{C}_Y^\infty\big)^{n_{i1}}$ and $\pi = \bigoplus_{i=1}^{d} \pi_i$. We have a surjective morphism $\pi\colon E \longrightarrow \mathcal{H}$. Let $\mathcal{N}_i = \ker \pi_i$ and $\mathcal{N} = \ker \pi$. Then we have an exact sequence

$$0 \longrightarrow \mathcal{N}_i \longrightarrow \mathcal{N}_{|U_i} \longrightarrow \bigoplus_{j \neq i} \big(\mathcal{C}_Y^\infty\big)^{n_{j1}} \longrightarrow 0.$$

By Lemma 7.10 $\mathcal{N}_{|U_i}$ is isomorphic to $\mathcal{N}_i \oplus \big(\mathcal{C}_Y^\infty\big)^{\sum_{j\neq i} n_{j1}}$. Furthermore $\mathcal{N}_i$ admits a finite free resolution of length $N-1$. Thus $\mathcal{N}$ admits a finite free resolution of length at most $N-1$ in a neighbourhood of every point in K. By induction $\mathcal{N}$ admits a finite global locally free resolution in a neighbourhood of K.

(ii) Let $\big(E_i\big)_{1\leq i\leq N}$ and $\big(F_i\big)_{1\leq i\leq N}$ be two finite locally free resolutions in a neighbourhood of K. Suppose that we have constructed $\big(G_i\big)_{1\leq i\leq k}$ and injections $E_\bullet \hookrightarrow G_\bullet$ and $F_\bullet \hookrightarrow G_\bullet$. Then we have two diagrams

$$\begin{tikzcd}
0 \arrow[r] & E_k \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d] & Q_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & E_{k-1} \arrow[r] \arrow[d] & G_{k-1} \arrow[r] \arrow[d] & Q_{k-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & E_1 \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & Q_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
0 \arrow[r] & F_k \arrow[r] \arrow[d] & G_k \arrow[r] \arrow[d] & R_k \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_{k-1} \arrow[r] \arrow[d] & G_{k-1} \arrow[r] \arrow[d] & R_{k-1} \arrow[r] \arrow[d] & 0 \\
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & F_1 \arrow[r] \arrow[d] & G_1 \arrow[r] \arrow[d] & R_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{H} \arrow[r] \arrow[d] & \mathcal{H} \arrow[r] \arrow[d] & 0 & \\
 & 0 & 0 & &
\end{tikzcd}$$

where all the lines and the columns are exact and $Q_1, \ldots, Q_k, R_1, \ldots, R_k$ are locally free. Let $N_k = \ker\big(E_k \to E_{k-1}\big)$, $N'_k = \ker\big(F_k \to F_{k-1}\big)$, $N''_k = \ker\big(G_k \to G_{k-1}\big)$, $\widetilde{Q}_k = \ker\big(Q_k \to Q_{k-1}\big)$ and $\widetilde{R}_k = \ker\big(R_k \to R_{k-1}\big)$. By breaking the exact sequences of the two last columns into short exact sequences, we obtain that $\widetilde{Q}_k$ and $\widetilde{R}_k$ are locally free. We have two exact sequences $0 \longrightarrow N_k \longrightarrow N''_k \longrightarrow \widetilde{Q}_k \longrightarrow 0$ and $0 \longrightarrow N'_k \longrightarrow N''_k \longrightarrow \widetilde{R}_k \longrightarrow 0$. By Lemma 7.10, $N''_k \simeq N_k \oplus \widetilde{Q}_k \simeq N'_k \oplus \widetilde{R}_k$, and we define $G_{k+1} = \big(E_{k+1} \oplus \widetilde{Q}_k\big) \oplus \big(F_{k+1} \oplus \widetilde{R}_k\big)$. We put $G_{N+1} = N''_N$ to end the resolution $G_\bullet$. □

Let $\mathcal{F}$ be a relatively coherent sheaf on $\big(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}, \pi, B\big)$, where $B = M/_G$. Then we have $\mathfrak{X} = \big(\mathfrak{X} \times_B M\big)/_G$ and $\mathfrak{X} \times_B M$ is smooth. We have seen that $\mathcal{F}^\infty$ admits locally finite $\mathcal{C}_{\mathfrak{X}}^\infty$-free resolutions. By Proposition 7.9 (i), $\mathcal{F}^\infty$ admits a global locally $\mathcal{C}_{\mathfrak{X}}^\infty$-free resolution $E_\bullet$ on any relatively compact open subset $U \subset\subset \mathfrak{X}$. By Proposition 7.9 (ii), the element $\sum_{i=1}^{N} (-1)^i \big[E_i\big] \in K(U)$ is independent of the chosen resolution $E_\bullet$. We will denote it by $\big[\mathcal{F}^\infty\big]$.

Definition 7.11. Let $\mathcal{F}$ and $\mathcal{G}$ be relatively coherent sheaves on $\mathfrak{X}$ and $u \in \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}}(\mathcal{F}, \mathcal{G})$. We say that u is a *strict morphism* if for every $x \in \mathfrak{X}$ we can choose trivializations $\phi^{-1}\mathcal{F} \simeq pr_1^{-1}\overline{\mathcal{F}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}$ and $\phi^{-1}\mathcal{G} \simeq pr_1^{-1}\overline{\mathcal{G}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}$ such that $u \in \mathrm{Hom}_{\mathcal{O}_Z}(\overline{\mathcal{F}}, \overline{\mathcal{G}})$.

Lemma 7.12. *The category of relatively coherent sheaves where morphisms are strict morphisms is an abelian category.*

Proof. We define the kernel, cokernel, image, coimage as the usual ones in the abelian category $\mathrm{Mod}(\mathcal{O}_{\mathfrak{X}^{\mathrm{rel}}}^{\mathrm{rel}})$. Let $u: \mathcal{F} \longrightarrow \mathcal{G}$ be a strict morphism. Then we can take trivializations

$$\phi^{-1}\mathcal{F} \simeq \mathrm{pr}_1^{-1}\overline{\mathcal{F}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}} \quad \text{and} \quad \phi^{-1}\mathcal{G} \simeq \mathrm{pr}_1^{-1}\overline{\mathcal{G}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}$$

such that $u \in \mathrm{Hom}_{\mathcal{O}_Z}(\overline{\mathcal{F}}, \overline{\mathcal{G}})$. By Lemma 7.7 (ii), $\mathcal{O}_{Z\times V}^{\mathrm{rel}}$ is flat over $pr_1^{-1}\mathcal{O}_Z$, so that

$$\phi^{-1}\ker\big(u: \mathcal{F} \longrightarrow \mathcal{G}\big) \simeq pr_1^{-1}\ker\big(u: \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}}\big) \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}$$

$$\phi^{-1}\mathrm{coker}\big(u: \mathcal{F} \longrightarrow \mathcal{G}\big) \simeq pr_1^{-1}\mathrm{coker}\big(u: \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}}\big) \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}$$

$$\phi^{-1}\mathrm{im}\big(u: \mathcal{F} \longrightarrow \mathcal{G}\big) \simeq pr_1^{-1}\mathrm{im}\big(u: \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}}\big) \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}$$

$$\phi^{-1}\mathrm{coim}\big(u: \mathcal{F} \longrightarrow \mathcal{G}\big) \simeq pr_1^{-1}\mathrm{coim}\big(u: \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}}\big) \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}$$

Thus $\ker u$, $\mathrm{coker}\, u$, $\mathrm{im}\, u$ and $\mathrm{coim}\, u$ are relatively coherent. Furthermore, the morphisms $\ker u \longrightarrow \mathcal{F}$, $\mathcal{F} \longrightarrow \mathrm{coim}\, u$, $\mathrm{im}\, u \longrightarrow \mathcal{G}$ and $\mathcal{G} \longrightarrow \mathrm{coker}\, u$ are strict morphisms. Finally, $\mathrm{coim}\, u \simeq \mathrm{im}\, u$ since $\mathrm{Mod}(\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}})$ is an abelian category. □

The associated Grothendieck group of this abelian category will be denoted by $K^{\mathrm{rel}}(\mathfrak{X})$.

Definition 7.13. Let $f: \mathfrak{X}/B \longrightarrow \mathfrak{X}'/B$ be a smooth map over B holomorphic in the fibers and $\mathcal{F}$, $\mathcal{G}$ be two relatively coherent sheaves for f on $\mathfrak{X}$. A morphism u in $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}}(\mathcal{F}, \mathcal{G})$ is *strict for f* if for all $x' \in \mathfrak{X}'$ there exist a trivialization of $\mathfrak{X}'$ in a neighbourhood of $U_{x'}$ of x' and a trivialization ϕ of $\mathfrak{X}$ in a neighbourhood of $f^{-1}(U_{x'}) \cap (\mathrm{supp}\,\mathcal{F} \cup \mathrm{supp}\,\mathcal{G})$ in which $f: Z \times V \longrightarrow Z' \times V'$ is of the form $(z, v) \longrightarrow (g(z), v)$ such that the morphism

$$pr_1^{-1}\overline{\mathcal{F}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}} \simeq \phi^{-1}\mathcal{F} \xrightarrow{\phi^{-1}u} \phi^{-1}\mathcal{G} \simeq pr_1^{-1}\overline{\mathcal{G}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}_{Z\times V}^{\mathrm{rel}}$$

belongs to $\mathrm{Hom}_{\mathcal{O}_Z}(\overline{\mathcal{F}}, \overline{\mathcal{G}})$.

The category of relatively coherent sheaves for f with strict morphisms for f is also an abelian category. We will denote it by $K_f^{\mathrm{rel}}(\mathfrak{X})$.

The usual operations still exist in relative K-theory and are listed in the following definition:

Definition 7.14.

(i) If $\mathcal{F}$ and $\mathcal{G}$ are relatively coherent sheaves on $\mathfrak{X}$, $\mathcal{F} . \mathcal{G}$ is defined in $K^{\mathrm{rel}}(\mathfrak{X})$ by

$$\mathcal{F} . \mathcal{G} = \sum_i (-1)^i \,\mathrm{Tor}^i_{\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}}(\mathcal{F}, \mathcal{G}).$$

(ii) If $\mathcal{F}$ is a relatively coherent sheaf on $\mathfrak{X}$, $\mathcal{F}^\vee$ is defined in $K^{\mathrm{rel}}(\mathfrak{X})$ by

$$\mathcal{F}^\vee = \sum_i (-1)^i \,\mathcal{E}xt^i_{\mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}}(\mathcal{F}, \mathcal{O}_{\mathfrak{X}}^{\mathrm{rel}}).$$

(iii) If $f\colon \mathfrak{X}/B \longrightarrow \mathfrak{X}'/B'$ satisfies the hypotheses of Lemma 7.3, then for every relatively coherent sheaf $\mathcal{F}$ on $\mathfrak{X}'$, $f^{\dagger}\mathcal{F}$ is defined in $K^{\mathrm{rel}}(\mathfrak{X})$ by

$$f^{\dagger}\mathcal{F} = \sum_{i\geq 0}(-1)^i \operatorname{Tor}^i_{f^{-1}\mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}'}}(f^{-1}\mathcal{F}, \mathcal{O}^{\mathrm{rel}}_{\mathfrak{X}}).$$

(iv) If $f\colon \mathfrak{X}/B \longrightarrow \mathfrak{X}'/B'$ and if $\mathcal{F}$ is relatively coherent for f, $f_!\mathcal{F}$ is defined in $K^{\mathrm{rel}}(\mathfrak{X}')$ by

$$f_!\mathcal{F} = \sum_{i\geq 0}(-1)^i R^i f_*\mathcal{F}.$$

The product in (i) endows $K^{\mathrm{rel}}(\mathfrak{X})$ with a structure of a unitary commutative ring. The map $f^{\dagger}$ in (ii) induces a ring morphism $f^{\dagger}\colon K^{\mathrm{rel}}(\mathfrak{X}') \longrightarrow K^{\mathrm{rel}}(\mathfrak{X})$. The map $f_!$ in (iii) induces a map $f_!\colon K^{\mathrm{rel}}_f(\mathfrak{X}) \longrightarrow K^{\mathrm{rel}}(\mathfrak{X}')$.

Proposition 7.15. *Let $\mathfrak{X}'$ be a relatively compact open subset of $\mathfrak{X}$. The map which associates to any relatively coherent sheaf $\mathcal{F}$ on $\mathfrak{X}$ the class $[\mathcal{F}^{\infty}]$ in $K(\mathfrak{X}')$ induces a ring morphism $K^{\mathrm{rel}}(\mathfrak{X}) \longrightarrow K(\mathfrak{X}')$.*

Proof. We must show that if $0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0$ is a strict exact sequence of relatively coherent sheaves, then $[\mathcal{F}^{\infty}]-[\mathcal{G}^{\infty}]+[\mathcal{H}^{\infty}] = 0$. This sequence is locally isomorphic to

$$0 \longrightarrow pr_1^{-1}\overline{\mathcal{F}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V} \longrightarrow pr_1^{-1}\overline{\mathcal{G}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V} \longrightarrow pr_1^{-1}\overline{\mathcal{H}} \otimes_{pr_1^{-1}\mathcal{O}_Z} \mathcal{O}^{\mathrm{rel}}_{Z\times V} \longrightarrow 0,$$

and is obtained by extension of the structure sheaves from an exact sequence of coherent analytic sheaves $0 \longrightarrow \overline{\mathcal{F}} \longrightarrow \overline{\mathcal{G}} \longrightarrow \overline{\mathcal{H}} \longrightarrow 0$ on Z. We can construct locally free resolutions $E_{\overline{\mathcal{F}},\bullet}$, $E_{\overline{\mathcal{G}},\bullet}$, $E_{\overline{\mathcal{H}},\bullet}$ of $\overline{\mathcal{F}}$, $\overline{\mathcal{G}}$, $\overline{\mathcal{H}}$ related by an exact sequence $0 \longrightarrow E_{\overline{\mathcal{F}},\bullet} \longrightarrow E_{\overline{\mathcal{G}},\bullet} \longrightarrow E_{\overline{\mathcal{H}},\bullet} \longrightarrow 0$. Using cut-off functions again, we patch these exact sequences together step by step and obtain resolutions $E_{\mathcal{F}^{\infty},\bullet}$, $E_{\mathcal{G}^{\infty},\bullet}$, $E_{\mathcal{H}^{\infty},\bullet}$ of $\mathcal{F}^{\infty}$, $\mathcal{G}^{\infty}$, $\mathcal{H}^{\infty}$ on $\mathfrak{X}'$ related by an exact sequence

$$0 \longrightarrow E_{\mathcal{F}^{\infty},\bullet} \longrightarrow E_{\mathcal{G}^{\infty},\bullet} \longrightarrow E_{\mathcal{H}^{\infty},\bullet} \longrightarrow 0.$$

□

We finally state a result which will allow us to move relative integrable structures without modifying the classes in K-theory.

Lemma 7.16. *Let $\mathcal{F}$ be a relative coherent sheaf on $\mathfrak{X}\times[0,1]/B\times[0,1]$, and for all t in $[0,1]$, let $i_t\colon \mathfrak{X}/B\times\{t\} \longrightarrow \mathfrak{X}\times[0,1]/B\times[0,1]$ be the canonical injection. Then for all $\mathfrak{X}' \subset\subset \mathfrak{X}$, $\left[i_t^*\mathcal{F}^{\infty}\right] \in K(\mathfrak{X}')$ is independent of t.*

Proof. We take a resolution $E_{\bullet}$ of $\mathcal{F}^{\infty}$ in $\mathfrak{X}'$. Then, since $\mathcal{F}$ is flat over $[0,1]$, $i_t^*E_{\bullet}$ is a resolution of $i_t^*\mathcal{F}^{\infty}$. This gives the result. □

8. Appendix II: the decomposition theorem for semi-small maps

In this appendix, we provide Le Potier's unpublished proof of the decomposition theorem for semi-small maps. For the formalism of the six operations in the derived category of constructible sheaves, we refer to [**Di**] and [**Ka-Sc**].

Let X be a complex irreductible quasi-projective variety endowed with a stratification X_ν. For $k \in \mathbb{N}$, we define $U_k = \bigsqcup_{\operatorname{codim}(X_\nu)\geq k} X_\nu$. The U_k's form an increasing family of open sets in X. For any constructible complex $C^\bullet$ on X, we define $C_k^\bullet = C^\bullet_{|U_k}$.

Let us recall the definition of the intersection cohomology ([**G0-McP**]).

Definition 8.1. Let $\mathcal{L}$ be a local system of $\mathbb{Q}$-vector spaces on U_0. The intersection complex $IC(\mathcal{L})$ associated to $\mathcal{L}$ with the middle perversity is a bounded constructible complex on X satisfying the following conditions:

(i) $IC(\mathcal{L})_0 \simeq \mathcal{L}$
(ii) $\mathcal{H}^i(IC(\mathcal{L})_0) = 0$ if $i > 0$
(iii) If $j \geq 1$ and $j \geq k$, $\mathcal{H}^j(IC(\mathcal{L})_k) = 0$
(iv) If $k \geq 1$ and $i\colon U_k \longrightarrow U_{k+1}$ is the canonical injection, then the morphism

$$I_{k+1} \longrightarrow R\,i_*\,i^{-1}\,I_{k+1} = R\,i_* I_k$$

is a quasi-isomorphism in degrees $\leq k$.

In the bounded derived category of $\mathbb{Q}$-constructible sheaves on X, $IC(\mathcal{L})$ is unique up to a unique isomorphism.

For any stratum S of codimension k in X, let $j_S\colon S \longrightarrow X$ be the corresponding inclusion. If $i\colon U_{k-1} \longrightarrow U_k$ is the canonical injection, we have the adjunction triangle

$$\bigoplus_{S,\,\operatorname{codim} S=k} j_{S*}\,j_S^!IC(\mathcal{L}) \longrightarrow IC(\mathcal{L})_k \longrightarrow R\,i_*\,IC(\mathcal{L})_{k-1} \xrightarrow[+1]{}$$

The conditions (iii) and (iv) imply that $\mathcal{H}^i\big(j_S^!IC(\mathcal{L})\big) = 0$ if $i \leq k$.

Let $f\colon Y \longrightarrow X$ be a semi-small map between complex irreducible algebraic varieties. Recall that X is stratified in such a way that if $Y_\nu = f^{-1}(X_\nu)$, $f_\nu = f_{|Y_\nu}\colon Y_\nu \longrightarrow X_\nu$ is a topological fibration. If d_ν is the maximal real dimension of an irreducible component of the fiber, the semi-smallness condition means that for all ν we have $\operatorname{codim}_X X_\nu \geq d_\nu$. By definition, the equality occurs for essential strata. The local systems $\mathcal{L}_\nu = R^{d_\nu} f_{\nu*}\mathbb{Q}_{Y_\nu}$ are the monodromy local systems on the strata X_ν, and $j_\nu\colon \overline{X}_\nu \longrightarrow X$ are the canonical injections. We suppose that Y is rationally smooth, which means that $\omega_Y \simeq \mathbb{Q}_Y[2\dim Y]$.

Theorem 8.2. [Decomposition theorem] *Under the hypotheses above, there exists a canonical quasi-isomorphism*

$$Rf_*\mathbb{Q}_Y \xrightarrow{\ \sim\ } \bigoplus_{\nu\ \textit{essential}} j_{\nu*}\,IC_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu].$$

Proof. The main tool of the proof is the following lifting lemma:

Lemma 8.3. *Let D be the bounded derived $\mathbb{Q}$-constructible category on X, $A^\bullet$, $B^\bullet$ and $C^\bullet$ three complexes in D and $f\colon B^\bullet \longrightarrow C^\bullet$ a morphism in D such that*

(i) $A^\bullet$ *is concentrated in degrees* $\leq k$,
(ii) f *induces in cohomology a morphism which is bijective in degrees* $\leq k-1$, *and injective in degree* k.

Then

(i) *The morphism* $\phi_f\colon Hom_D(A^\bullet, B^\bullet) \longrightarrow Hom_D(A^\bullet, C^\bullet)$ *induced by* f *is injective.*
(ii) *The image of* ϕ_f *consists of morphisms* $g\colon A^\bullet \longrightarrow C^\bullet$ *in* D *such that the induced morphism* $\mathcal{H}^k(A^\bullet) \longrightarrow \mathcal{H}^k(C^\bullet)$ *factors through* $\mathcal{H}^k(B^\bullet)$.

Remark 8.4. When f induces in cohomology a bijective morphism in degrees $\leq k$, this is the proposition of page 95 of [**G0-McP**].

Proof. Let $M^\bullet$ be the mapping cone of f. From the distinguished triangle

$$B^\bullet \longrightarrow C^\bullet \longrightarrow M^\bullet \xrightarrow[+1]{}$$

and the hypotheses, $\mathcal{H}^q(M^\bullet) = 0$ for $q \leq k-1$. Therefore $\mathrm{Hom}_D(A^\bullet, M^\bullet[-1]) = 0$. Now, we have a distinguished triangle

$$\mathrm{RHom}_D(A^\bullet, B^\bullet) \longrightarrow \mathrm{RHom}_D(A^\bullet, C^\bullet) \longrightarrow \mathrm{RHom}_D(A^\bullet, M^\bullet) \xrightarrow[+1]{}$$

This gives the long exact sequence

$$\mathrm{Hom}_D(A^\bullet, M^\bullet[-1]) \longrightarrow \mathrm{Hom}_D(A^\bullet, B^\bullet) \xrightarrow{\phi_f} \mathrm{Hom}_D(A^\bullet, C^\bullet) \longrightarrow \mathrm{Hom}_D(A^\bullet, M^\bullet)$$

which proves (i).

For (ii), remark that $M^\bullet$ is concentrated in degrees $\geq k$ and that $A^\bullet$ is concentrated in degrees $\leq k$, so that $\mathrm{Hom}_D(A^\bullet, M^\bullet) \simeq \mathrm{Hom}_D(\mathcal{H}^k(A^\bullet), \mathcal{H}^k(M^\bullet))$. Thus

$$\mathrm{Im}\,\phi_f = \big\{ g \in \mathrm{Hom}_D(A^\bullet, C^\bullet) \;\big/\; \text{the induced morphism} \quad \mathcal{H}^k(A^\bullet) \longrightarrow \mathcal{H}^k(M^\bullet) \;\; \text{vanishes} \big\}\cdot$$

From the exact sequence $0 \longrightarrow \mathcal{H}^k(B^\bullet) \longrightarrow \mathcal{H}^k(C^\bullet) \longrightarrow \mathcal{H}^k(M^\bullet)$, we get the result. □

We now turn to the proof of the decomposition theorem. The quasi-isomorphism

$$Rf_*\mathbb{Q}_Y \xrightarrow{\sim} \bigoplus_{\nu \text{ essential}} j_{\nu *}\, IC_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu]$$

will be constructed by induction on the increasing family of open sets U_i associated to the stratification on X.

On U_0, the quasi-isomorphism reduces to $Rf_{0*}\,\mathbb{Q}_{Y_0} \simeq \mathcal{L}_0$, which is the definition of $\mathcal{L}_0$.

Suppose now that we have constructed a quasi-isomorphism

$$\lambda_{k-1} : \big(Rf_*\mathbb{Q}_Y\big)_{k-1} \xrightarrow{\sim} \bigoplus_{\substack{\nu \text{ essential} \\ d_\nu \leq k-1}} j_{\nu *}\, IC_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu]$$

We introduce the following notations:

(i) $\mathcal{S} = \displaystyle\bigoplus_{\substack{\nu \text{ essential} \\ d_\nu \leq k-1}} j_{\nu *}\, IC_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu]$,
(ii) $i\colon U_{k-1} \longrightarrow U_k$ is the injection,

(iii) $A^\bullet = \big(Rf_*\mathbb{Q}_Y\big)_k$, $\quad B^\bullet = \mathcal{S}_k$, $\quad C^\bullet = R\,i_*\mathcal{S}_{k-1}$,

(iv) $f\colon B^\bullet \longrightarrow C^\bullet$ is the adjunction morphism $\mathcal{S}_k \longrightarrow R\,i_* i^{-1}\mathcal{S}_k = R\,i_*\,\mathcal{S}_{k-1}$.

We have a morphism $g\colon A^\bullet \longrightarrow C^\bullet$ given by the chain of morphisms

$$\big(Rf_*\mathbb{Q}_Y\big)_k \longrightarrow R\,i_* i^{-1}\big(Rf_*\mathbb{Q}_Y\big)_k = R\,i_*\big(Rf_*\mathbb{Q}_Y\big)_{k-1} \xrightarrow{R\,i_*\lambda_{k-1}} R\,i_*\,\mathcal{S}_{k-1}$$

We now check the hypotheses of the lifting lemma.

— $\big(Rf_*\mathbb{Q}_Y\big)_k$ is concentrated in degrees $\leq k$ since the fibers of f over U_k have real dimension $\leq k$, and by base change, for every x in X, $\big(Rf_*\mathbb{Q}_Y\big)_x = R\Gamma\big(f^{-1}(x), \mathbb{Q}\big)$.

— Let S be a stratum of codimension k in X and X_ν an essential stratum with $d_\nu \leq k-1$. Then $S \neq X_\nu$ and we have either $S \subsetneq X_\nu$ or $S \cap X_\nu = \emptyset$ (which is irrelevant). Let $j_S\colon S \longrightarrow X$ be the injection of the stratum S in X and $j_{S,\nu}\colon S \longrightarrow \overline{X}_\nu$ be the injection of S in $\overline{X}_\nu$. The cartesian diagram

$$\begin{tikzcd} S \arrow[r, "\mathrm{id}"] \arrow[d, "j_{S,\nu}"'] & S \arrow[d, "j_S"] \\ \overline{X}_\nu \arrow[r, "j_\nu"'] & X \end{tikzcd}$$

gives $j_S^!\, j_{\nu*}\, IC_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu] = j_{S,\nu}^!\, IC_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu]$. We have $\mathcal{H}^q\big(j_{S,\nu}^!\, IC_{\overline{X}_\nu}(\mathcal{L}_\nu)[-d_\nu]\big) = 0$ if $q - d_\nu \leq \operatorname{codim}_{\overline{X}_\nu} S = k - d_\nu$, i.e. $q \leq k$. Thus $\mathcal{H}^q(j_S^!\,\mathcal{S}) = 0$ for $q \leq k$.

Let us now write the adjunction triangle $j_{S*}\, j_S^!\,\mathcal{S}_k \longrightarrow \mathcal{S}_k \xrightarrow{f} R\,i_{S*}\, i_S^{-1}\mathcal{S}_k \xrightarrow[+1]{}$. The previous result proves that f is a quasi-isomorphism in degrees $\leq k-1$ on S, and then on U_k. In degree k, since $\mathcal{H}^k\big(j_{S*}\, j_S^!\,\mathcal{S}_k\big) = 0$, the induced map $\mathcal{H}^k(f)$ is injective.

We can also remark that $\mathcal{H}^k(\mathcal{S}_k) = 0$. Indeed, on $U_k \cap \overline{X}_\nu$, all strata have codimension at most $k - d_\nu$, so that $\mathcal{H}^j\big(IC_{\overline{X}_\nu}(\mathcal{L}_\nu)_{|U_k \cap \overline{X}_\nu}\big) = 0$ if $j \geq k - d_\nu$.

— We want to lift g to a morphism $\widetilde{\lambda}_k\colon A^\bullet \longrightarrow B^\bullet$. Since $\mathcal{H}^k(B^\bullet) = 0$, the condition (ii) of the lifting lemma means that the map $\mathcal{H}^k(g)$ vanishes.

We will prove a stronger result, namely that the map $\theta : \big(R^k\, f_*\,\mathbb{Q}_Y\big)_k \longrightarrow R^k\, i_*\big(Rf_*\,\mathbb{Q}_Y\big)_{k-1}$ vanishes. Let $F_k = U_k \setminus U_{k-1}$ be the closed set consisting of all k-codimensional strata in X and let $j : F_k \longrightarrow U_k$ be the inclusion. The vanishing of θ is equivalent to the surjectivity of the map

$$\psi : \mathcal{H}^k\big(j_*\, j^!\, Rf_*\,\mathbb{Q}_Y\big) \longrightarrow \mathcal{H}^k\big(Rf_*\,\mathbb{Q}_Y\big)_k$$

Let X_ν be a stratum in F_k. We consider the following cartesian diagram:

$$\begin{tikzcd} Y_\nu \arrow[r, "i_\nu"] \arrow[d, "f_\nu"'] & Y \arrow[d, "f"] \\ X_\nu \arrow[r, "j_\nu"'] & X \end{tikzcd}$$

If D is the Verdier duality functor, then

$$\begin{aligned} j_\nu^!\, Rf_*\, \mathbb{Q}_Y &= Rf_{\nu *}\, i_\nu^!\, \mathbb{Q}_Y = Rf_{\nu *}\, i_\nu^!\, \omega_Y[-2\dim Y] && \text{by hypothesis} \\ &= Rf_{\nu *}\, \omega_{Y_\nu}[-2\dim Y] = D\big(Rf_{\nu *}\, \mathbb{Q}_{Y_\nu}\big)[-2\dim Y] \\ &= \mathcal{RH}om_{\mathbb{Q}_{X_\nu}}\big(Rf_{\nu *}\, \mathbb{Q}_{Y_\nu}, \mathbb{Q}_{X_\nu}\big)[2\dim X_\nu - 2\dim Y] && \text{since } X_\nu \text{ is smooth.} \end{aligned}$$

Now $\mathcal{H}^k\big(j_\nu^!\, Rf_*\, \mathbb{Q}_Y\big) = \mathcal{H}om_{\mathbb{Q}_{X_\nu}}\big(R^{2\dim Y - 2\dim X_\nu - k} f_{\nu *}\, \mathbb{Q}_{Y_\nu}, \mathbb{Q}_{X_\nu}\big)$. Since $k = \dim X - \dim X_\nu$, we obtain

$$\mathcal{H}^k\big(j_\nu^!\, Rf_*\, \mathbb{Q}_Y\big) = \mathcal{H}om_{\mathbb{Q}_{X_\nu}}\big(R^k f_{\nu *}\, \mathbb{Q}_{Y_\nu}, \mathbb{Q}_{X_\nu}\big) = \begin{cases} \mathcal{L}_\nu^* & \text{if } X_\nu \text{ is essential} \\ 0 & \text{otherwise} \end{cases}$$

Remark that the fibers of f_ν are projective varieties, so that $\mathcal{L}_\nu \simeq \mathcal{L}_\nu^*$. This gives

$$\mathcal{H}^k\big(j_*\, j^{\,!}\, Rf_*\, \mathbb{Q}_Y\big) \simeq \bigoplus_{\substack{\nu \text{ essential} \\ \operatorname{codim}(X_\nu) = k}} j_{\nu *}\mathcal{L}_\nu.$$

This isomorphism can be interpreted in the following way: if we consider the canonical morphism $\big(Rf_*\, \mathbb{Q}_Y\big)_k \longrightarrow \big(R^k f_*\, \mathbb{Q}_Y\big)_k[-k]$, then $\mathcal{H}^k_{F_k}\big(Rf_*\, \mathbb{Q}_Y\big)_k \longrightarrow \mathcal{H}^k_{F_k}\big(R^k f_*\, \mathbb{Q}_Y[-k]\big)_k$ is a quasi-isomorphism. Therefore, in the following diagram,

$$\begin{tikzcd} \mathcal{H}^k_{F_k}\big(Rf_*\, \mathbb{Q}_Y\big)_k \arrow[r, "\sim"] \arrow[d, "\psi"'] & \mathcal{H}^0_{F_k}\big(R^k f_*\, \mathbb{Q}_Y\big)_k \arrow[d, "\sim"] \\ \mathcal{H}^k\big(Rf_*\, \mathbb{Q}_Y\big)_k \arrow[r, "\sim"] & \big(R^k f_*\, \mathbb{Q}_Y\big)_k \end{tikzcd}$$

the map ψ is an isomorphism, in particular, it is surjective. So we can apply the lifting lemma and we get a canonical morphism $\widetilde{\lambda}_k \colon \big(Rf_*\, \mathbb{Q}_Y\big)_k \longrightarrow \mathcal{S}_k$ such that the diagram

$$\begin{tikzcd} \big(Rf_*\, \mathbb{Q}_Y\big)_k \arrow[r] \arrow[d, "\widetilde{\lambda}_k"'] & R\, i_*\big(Rf_*\, \mathbb{Q}_Y\big)_{k-1} \arrow[d, "R\, i_*\, \lambda_{k-1}"] \\ \mathcal{S}_k \arrow[r, "f"'] & R\, i_*\, \mathcal{S}_{k-1} \end{tikzcd}$$

commutes. We look now at the following morphism of distinguished triangles:

$$\begin{tikzcd} \displaystyle\bigoplus_{\nu,\, c_\nu = k} j_{\nu *}\, j_\nu^!\big(Rf_*\, \mathbb{Q}_Y\big)_k \arrow[r] \arrow[d, "j_{\nu *}\, j_\nu^!\, \widetilde{\lambda}_k"'] & \big(Rf_*\, \mathbb{Q}_Y\big)_k \arrow[r] \arrow[d, "\widetilde{\lambda}_k"'] & R\, i_*\big(Rf_*\, \mathbb{Q}_Y\big)_{k-1} \arrow[r, "+1"] \arrow[d, "R\, i_*\, \lambda_{k-1}", "\simeq"'] & {} \\ \displaystyle\bigoplus_{\nu,\, c_\nu = k} j_{\nu *}\, j_\nu^!\, \mathcal{S}_k \arrow[r] & \mathcal{S}_k \arrow[r] & R\, i_*\, \mathcal{S}_{k-1} \arrow[r, "+1"'] & {} \end{tikzcd}$$

where $c_\nu = \operatorname{codim}_X(X_\nu)$.

— Since $j_\nu^!\big(Rf_*\mathbb{Q}_Y\big)_k \simeq \mathcal{RH}om_{\mathbb{Q}_{X_\nu}}\big(Rf_*\mathbb{Q}_{Y_\nu}, \mathbb{Q}_{X_\nu}\big)[-2k]$, the complex $j_\nu^!\big(Rf_*\mathbb{Q}_Y\big)_k$ is concentrated in degrees $\geq 2k - d_\nu$. The semi-smallness of f implies $k = c_\nu \geq d_\nu$. Thus $j_\nu^!\big(Rf_*\mathbb{Q}_Y\big)_k$ is concentrated in degrees $\geq k$.

— The complex $j_\nu^!\, \mathcal{S}_k$ is concentrated in degrees $\geq k + 1$.

This shows that $\widetilde{\lambda}_k$ is a quasi-isomorphism in degrees $\leq k-2$.

If we denote by $A \longrightarrow B \longrightarrow C \xrightarrow{+1}$ and $A' \longrightarrow B' \longrightarrow C' \xrightarrow{+1}$ the two distinguished triangle above, we get a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{H}^{k-1}(B) \arrow[r] \arrow[d] & \mathcal{H}^{k-1}(C) \arrow[r] \arrow[d, "\simeq"] & \mathcal{H}^{k}(A) \arrow[r] \arrow[d] & \mathcal{H}^{k}(B) \\ 0 \arrow[r] & \mathcal{H}^{k-1}(B') \arrow[r] & \mathcal{H}^{k-1}(C') \arrow[r] & 0 & \end{tikzcd}$$

We have seen that the map $\mathcal{H}^k(A) \longrightarrow \mathcal{H}^k(B)$ is a quasi-isomorphism. This implies that $\mathcal{H}^{k-1}(B) \simeq \mathcal{H}^{k-1}(C)$ and proves that $\widetilde{\lambda}_k$ is a quasi-isomorphism in degree $k-1$.

Let $\mu_k \colon \big(Rf_* \mathbb{Q}_Y\big)_k \longrightarrow \big(R^k f_* \mathbb{Q}_Y\big)_k[-k] = \bigoplus_{\substack{\nu \text{ essential} \\ d_\nu = k}} j_{\nu *} \mathcal{L}_\nu[-k]$. We define

$$\lambda_k = \big(\widetilde{\lambda}_k, \mu_k\big) : \big(Rf_* \mathbb{Q}_Y\big)_k \longrightarrow \bigoplus_{\substack{\nu \text{ essential} \\ d_\nu \leq k}} j_{\nu *} \mathcal{L}_\nu[-k].$$

Then λ_k is a quasi-isomorphism in degrees $\leq k-1$ and also in degree k. It is zero in degrees $\geq k$. Therefore λ_k is a quasi isomorphism and the induction step is completed. This finishes the proof of the decomposition theorem.

$\square$

Bibliographie

[Ad-Le-Ru] A. Adem, J. Leida, Y. Ruan, Orbifolds and Stringy Topology, Cambridge University Press (2007).

[At-Hi] M. F. Atiyah, F. Hirzebruch, The Riemann Roch theorem for analytic imbeddings, Topology, *I*, (1962) 151–166.

[Br] J. Briançon, Description de $\mathrm{Hilb}^n\mathbb{C}\{x, y\}$, Inventiones math., *41*, (1977), 45–89.

[Ch-Ru] W. Chen, Y. Ruan, A new cohomology theory of orbifold, Comm. Math. Phys., *248:1*, (2004), 1–31.

[De-Be-Be-Ga] P. Deligne, A. Beilinson, J. Bernstein, O. Gabber, Faisceaux pervers, Astérisque, Soc. Math. France, *100*, (1983).

[Ch] J. Cheah, Cellular decompositions for various nested Hilbert schemes of points, Pac. J. Math. *183* (1998) 39–90.

[Da] G. Danila, Sur la cohomologie d'un fibré tautologique sur le schéma de Hilbert d'une surface, J. Algebraic Geom. *10*, (2001), 247–280.

[Di] A. Dimca, Sheaves in topology, Springer, (2004).

[Do] S. Donaldson, Symplectic submanifolds and almost-complex geometry, J. Diff. Geom., *44*, (1996), 666–705.

[El-Gö-Le] G. Ellingsrud, L. Göttsche, M. Lehn, On the cobordism class of the Hilbert scheme of a surface, J. Algebraic Geom., *10:1*, (2001), 81–100.

[El-St] G. Ellingsrud, S.-A. Strømme, On the cohomology of the Hilbert schemes of points in the plane, Invent. Math., *87*, (1987), 343–352.

[Fa-Gö] B. Fantechi, L. Göttsche, Orbifold cohomology for global quotients, Duke Math. J., *117:2*, (2003), 197–227.

[Fo] J. Fogarty, Algebraic families on an algebraic surface, Am. J. Math., *10*, (1968), 511–521.

[Fu] W. Fulton, Intersection Theory, Springer, (1984).

[G0-McP] M. Goresky, R. MacPherson Intersection homology theory II, Invent. Math., *73*, (1983), 77–129.

[Gö 1] L. Göttsche, The Betti numbers of the Hilbert scheme of points on a smooth projective surface, Math. Ann, *286*, (1990), 193–207.

[Gö 2] L. Göttsche, Hilbert Schemes of points on surfaces, ICM, *III 1–3*, (2002), 483–494.

[Gö-So] L. Göttsche, W. Soergel, Perverse sheaves and the cohomology of Hilbert schemes on smooth algebraic surfaces, Math. Ann, *296*, (1993), 235–245.

[Gr] I. Grojnowski, Instantons and affine algebras I: the Hilbert scheme and vertex operators Math. Res. Letters, *3*, (1996), 275–291.

[Ja-Ka-Ki] T. J. Jarvis, R. Kaufmann, T. Kimura, Stringy K-theory and the Chern character Invent. Math., *168*, (2007), 23-81.

[Ka-Sc] M. Kashiwara, P. Schapira, Sheaves on manifolds, Springer, (2002).

[Le] M. Lehn, Chern classes of tautological sheaves on Hilbert schemes of points on sufaces, Invent. Math., *136*, (1999), 157–207.

[Le-So-1] M. Lehn, C. Sorger Symmetric groups and the cup product on the cohomology of Hilbert schemes, Duke Math. J., *110:2*, (2001), 345–357.

[Le-So-2] M. Lehn, C. Sorger The cup product of Hilbert schemes for $K3$ surfaces, Invent. Math., *152:2*, (2003), 305–329.

[LP] J. Le Potier, La formule de Göttsche, Unpublished lecture notes.

[Li-Qi-Wa-1] W.-P. Li, Z. Qin, W. Wang, Generators for the cohomology ring of Hilbert schemes of points on surfaces, Internat. Math. Res. Notices, *20*, (2001), 1057–1074.

[Li-Qi-Wa-2] W.-P. Li, Z. Qin, W. Wang, Vertex algebras and the cohomology ring structure of Hilbert schemes of points on surfaces, Math. Ann., *324:1*, (2002), 105–133.

[Li-Qi-Wa-3] W.-P. Li, Z. Qin, W. Wang, Stability of the cohomology rings of Hilbert schemes of points on surfaces, Reine Angew. Math., *554*, (2003), 217–234.

[Ma] B. Malgrange, Division des distributions, Séminaire Schwartz, 1959–60, Exposé 25.

[Mi] J. Milnor, On the cobordism ring Ω^* and a complex analogue, Am. J. Math, *82*, (1960), 505–521.

[Na] H. Nakajima, Heisenberg algebra and Hilbert schemes on projective surfaces, Ann. Math, *145*, (1997), 379–388.

[No] S. Novikov, Topological invariance of rational Pontrjagin classes (Russian), Dokl. Akad. Nauk S. S. S. R., *163*, (1965), 298–300.

[Qi-Wa] Z. Qin, W Wang, Hilbert schemes and symmetric products: a dictionary, *Orbifolds in Mathematics and Physics (Madison, WI, 2001)*, Contemp. Math., *310*, Americam Mathematical Society (2002), 233–257.

[Ti] A. Tikhomirov, The variety of complex pairs of zero-dimensional subschemes of an algebraic surface, Izvestiya RAN: Ser. Mat *61*, (1997) 153–180. = Izvestiya: Mathematics, *61* (1997), 1265–1291.

[Vo 1] C. Voisin, On the Hilbert scheme of points on an almost complex fourfold, Ann. Inst. Fourier, *50,2*, (2000), 689–722.

[Vo 2] C. Voisin, On the punctual Hilbert scheme of a symplectic fourfold, Contemporary Mathematics, (2002), 265–289.

[Vo 3] C. Voisin, Théorie de Hodge et géométrie algébrique complexe, Société Mathématique de France, (2002).